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A Method of Moments Applied to an Invariant Imbedding Solution of a Certain Class of Fredholm Integral Equations

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A Method of Moments Applied to an Invariant Imbedding Solution of a Certain Class of Fredholm Integral Equations

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by

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This report is derived from a dissertation submitted to the Department of Mathematics and Statistics of The University of New Mexico in partial fulfillment of requirements for the Degree of Doctor of Philosophy.

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ABSTRACT

We consider the Fredholm integral equation

$$\phi(z) = g(z) + \int_{y}^{x} K(z,z') \, \phi(z') \, dz'$$

where g(z) represents a fairly large class of functions and

$$K(z,z') = \gamma(z') \int_{0}^{\infty} k(s') \, e^{-a(s')|z-z'|} \, ds' \quad .$$

This dissertation first develops a method for solving the integral equation when $\gamma(z')$ is a constant and then extends it to the case where $\gamma(z')$ is a step function. The solution of the integral equation is achieved by solving the integro differential invariant imbedding equations derived from the integral equation by varying the limits of integration. The imbedding equations are solved using a moment method which reduces the calculation to an initial value problem. Proofs of the existence and convergence of the method are given. In the case where $\gamma(z')$ is a constant, the solution of the integral equation is obtained by a simple quadrature of the product of the solution with a transform of the function g(z). In the case where $\gamma(z')$ is a step function, the integro differential equations for the reflection and transmission kernels are reduced to initial value problems and solved. These kernels are used to obtain fluxes from which the solution to the integral equation can again be obtained by simple quadratures.

Numerical examples are presented.

I. INTRODUCTION

We are concerned in this dissertation with the solution of Fredholm integral equations of the general form

$$\phi(z) = g(z) + \int_{y}^{x} K(z,z') \, \phi(z') \, dz' \qquad (1.1a)$$

where the kernel K has the integral representation

$$K(z,z') = \gamma(z') \int_{0}^{\infty} k(s') \, e^{-a(s')|z-z'|} \, ds' \quad . \qquad (1.1b)$$

It can be shown [1], [11] that this problem is equivalent to the "pseudo-transport" problem

$$\mathrm{sgn}(s) \frac{\partial}{\partial z} N(z,s) + a(s) \, N(z,s) = k(s) \, \gamma(z) \int_{-\infty}^{\infty} N(z,s') \, ds' \quad ,$$

$$y \le z \le x \quad , \qquad (1.2a)$$

$$N(y,s) = h(s) \quad , \qquad 0 < s < \infty \quad , \qquad (1.2b)$$

$$N(x,s) = f(s) \quad , \qquad -\infty < s < 0 \quad . \qquad (1.2c)$$

G. M. Wing [11] applied the method of invariant imbedding to equations (1.2) to arrive at a set of coupled non-linear partial differential integro equations. These equations form an initial value problem. Wing pointed out that the pair of solution functions for the imbedding equations together form a Green's function for equations (1.1). The solution of the imbedding equations is complicated by the fact that the equations themselves contain the solution evaluated at the end-points of the interval of interest, i.e. at y and x. If these end-point values are known, Wing's equations can be integrated.

By appropriately specializing Wing's equations one can derive a set of equations similar to the X and Y equations of Chandrasekhar [4] for the solutions at the end points. Under the assumptions that $y = -x$ and $\gamma(z)$ is an even function, these equations reduce to a set of ordinary differential integro equations. R. C. Allen [2] applied a method of moments to solve these reduced equations. The moment method results in a doubly infinite coupled set of ordinary differential equations. Existence and uniqueness of solutions to this set are provided by Allen and Kyner [3].

In this paper we carry out the invariant imbedding procedure to arrive at the general imbedding equations and the specialized imbedding equations for the solutions at the end points. Defining moments for these two sets of equations, we arrive at two new sets of equations, the imbedding moment equations and the specialized imbedding moment equations. Relationships between the moments are found and used to simplify the specialized imbedding moment equations. Other relationships connect the moments to the Taylor coefficients for the resolvent kernel of equation (1.1). Specializing to the case $\gamma(z') =$ constant, allows the reduction of the specialized imbedding moment and imbedding moment equations to equations which require integration in only a single variable. The reduced specialized imbedding moment equations are shown to satisfy the hypotheses of Allen and Kyner's existence and uniqueness theorem and so existence and uniqueness of their solutions is established. An existence and uniqueness proof is provided for the reduced imbedding moment equations.

Returning to the general case, integral reflection and transmission operators are defined and it is shown that the kernels of these operators satisfy integro differential equations. Using integral identities, these equations are reduced to ordinary differential equations which can be integrated simultaneously with the specialized imbedding equations when $\gamma(z')$ is a constant. The reflection and transmission kernels are used to obtain the solution for the case when $\gamma(z')$ is a step function.

Computational examples are presented to illustrate the method.

II. DERIVATION OF THE IMBEDDING EQUATIONS

We start by considering the equivalence between the integral equation (1.1) and the "pseudo-transport" problem (1.2). Specifically we consider

$$\operatorname{sgn}(s) \frac{\partial}{\partial z} N(z,s) + a(s)\, N(z,s) = k(s)\, \gamma(z) \int_{-\infty}^{\infty} N(z,s')\, ds' \quad ,$$

$$Y \le y \le z \le x \le X \quad , \qquad |s| < \infty \quad , \qquad (2.1a)$$

$$N(y,s) = h(s) \quad , \qquad 0 < s < \infty \quad , \qquad (2.1b)$$

$$N(x,s) = f(s) \quad , \qquad -\infty < s < 0 \quad , \qquad (2.1c)$$

where

R1 $k(s)$ is an even piecewise continuous function belonging to L_1;

R2 $a(s)$ is an even piecewise continuous function;

R3 either $a(s)$ or $k(s)$ is zero outside a finite interval or

$$\int_0^{\infty} |k(s')\, a^j(s')|\, ds' \le \frac{j!}{(X-Y)^j}\, r_j \text{ where } \sum_{j=0}^{\infty} r_j < \infty \quad ;$$

R4 Re $a(s) \ge 0$ for sufficiently large s;

R5 y and x lie in some fixed interval i.e. $Y \le y \le z \le x \le X$;

R6 $\gamma(z)$ is positive and piecewise continuous for y,z,x in the fixed interval $[Y,X]$;

R7 $f(s)$ and $h(s)$ are continuous functions with compact support on $-\infty < s < 0$ and $0 < s < \infty$ respectively;

R8 $f(s)$ and $h(s)$ are not both identically zero;

R9 The eigenvalues of the operator $\hat{K}$ defined by

$$\hat{K} \cdot = \int_y^x K(z,z') \,.\, dz' \quad , \qquad (2.2)$$

where K(z,z′) is given by equation (1.b), have absolute value greater than unity on $Y \le y \le x \le X$.

Assume that (2.1) has a unique solution, N(z,s), which is piecewise continuously differentiable with respect to z in $y \le z \le x$, and uniformly integrable with respect to s for $-\infty < s < \infty$. With

$$\eta(z) = \int_{-\infty}^{\infty} N(z,s') \, ds' \quad , \qquad (2.3)$$

equation (2.1) can be written in the form

$$\frac{\partial}{\partial z}\left[N(z,s)\ e^{a(s)z}\right] = k(s)\ \gamma(z)\ \eta(z)\ e^{a(s)z} \text{ for } s > 0 \quad , \qquad (2.4)$$

and

$$\frac{\partial}{\partial z}\left[-N(z,s)\ e^{-a(s)z}\right] = k(s)\ \gamma(z)\ \eta(z)\ e^{-a(s)z} \text{ for } s < 0 \quad . \qquad (2.5)$$

Using conditions (2.1b) and (2.1c) we carry out the integration and find

$$N(z,s) = k(s) \int_{y}^{z} \gamma(z')\ \eta(z')\ e^{a(s)(z'-z)}\ dz' + h(s)\ e^{a(s)(y-z)}$$

$$\text{for } s > 0 \qquad (2.6)$$

and

$$N(z,s) = k(s) \int_{z}^{x} \gamma(z')\ \eta(z')\ e^{a(s)(z-z')}\ dz' + f(s)\ e^{a(s)(z-x)}$$

$$\text{for } s < 0 \quad . \qquad (2.7)$$

Substituting equations (2.6) and (2.7) into equation (2.3) we arrive at

$$\eta(z) = \int_{-\infty}^{0} k(s') \, ds' \int_{z}^{x} \gamma(z') \, \eta(z') \, e^{a(s')(z-z')} \, dz'$$

$$+ \int_{0}^{\infty} k(s') \, ds' \int_{y}^{z} \gamma(z') \, \eta(z') \, e^{a(s')(z'-z)} \, dz'$$

$$+ \int_{-\infty}^{0} f(s') \, e^{a(s')(z-x)} \, ds' + \int_{0}^{\infty} h(s') \, e^{a(s')(y-z)} \, ds' \, . \quad (2.8)$$

At this point we would like to change the order of integration in the first two integrals in the right member of (2.8). To justify this we need to show that the integrand is absolutely integrable since then we can invoke Tonelli's theorem. To this end we note the following. By the piecewise continuity of $N(z,s)$ with respect to z and its uniform integrability with respect to s, $\eta(z)$ is a bounded function on the finite interval [Y,X]. Since $\gamma(z)$ is piecewise continuous on [Y,X], it is bounded there. The exponential is bounded, for in the first integral the fact that $\left| e^{a(s)(z-z')} \right| = e^{-\mathrm{Re}\, a(s) \, |z-z'|}$ and that $a(s)$ is piecewise continuous implies it is bounded for bounded s and restriction R4 implies its boundedness for sufficiently large s. A similar argument holds for the exponential in the second integral. From this we see that

$$\left| k(s) \, \gamma(z') \, e^{-a(s) \, |z-z'|} \right| < M \, |k(s)|$$

and absolute integrability follows from restriction R2.

Changing the order of integration and recalling that $a(s)$ and $k(s)$ are even we arrive at

$$\eta(z) = g(z) + \int_y^x \gamma(z') \, \eta(z') \, dz' \int_0^\infty k(s') \, e^{-a(s')|z-z'|} \, ds' \quad , \quad (2.9)$$

where

$$g(z) = \int_{-\infty}^0 f(s') \, e^{a(s')(z-x)} \, ds' + \int_0^\infty h(s') \, e^{a(s')(y-z)} \, ds' \quad . \quad (2.10)$$

Equation (2.9) is identical to (1.1) when the kernel is defined by equation (1.1b). This is the basis of the equivalence between the problems (1.1) and (2.1). We state this as a theorem.

<u>Theorem 2.1</u> Let $N(z,s)$ be piecewise continuously differentiable in z, $y \le z \le x$, uniformly integrable in s, $|s| < \infty$, and let $N(z,s)$ satisfy the "pseudo-transport" problem, (2.1). Then $\eta(z)$ given by (2.9) and (2.10) is a solution to (1.1). Conversely, if $\eta(z)$ is a solution to (1.1), then

$$N(z,s) = \begin{cases} k(s) \displaystyle\int_y^z \gamma(z') \, \eta(z') \, e^{a(s)(z'-z)} \, dz' + h(s) \, e^{a(s)(y-z)} \, , & s > 0 \\ k(s) \displaystyle\int_z^x \gamma(z') \, \eta(z') \, e^{a(s)(z-z')} \, dz' + f(s) \, e^{a(s)(z-x)} \, , & s < 0 \end{cases} \qquad (2.11)$$

is a solution to (2.1). This correspondence is unique.

Proof: The proof requires only minor modifications to that given in reference [11]. ∎

We now derive the invariant imbedding equations. This converts the solution of the integral equation (1.1) into the solution of an initial value problem and a simple quadrature. In the invariant imbedding approach the limits on the integral appearing in equation (1.1) are varied. This amounts to varying the boundary surfaces of the imaginary slab in the "pseudo-transport" problem (2.1). In order to make this dependence on the limits of integration more obvious, we will hereafter show the dependence explicitly. In our new notation the integral equation (1.1) becomes

$$\phi(z,x,y) = g(z) + \int_y^x K(z,z') \, \phi(z',x,y) \, dz' \quad . \quad (2.12)$$

The general theory of Fredholm integral equations, along with the condition R9, assert the existence of a resolvent kernel, $Q(z,z',x,y)$, such that the solution of equation (2.12) is given by:

$$\phi(z,x,y) = g(z) + \int_y^x Q(z,z',x,y) \, g(z') \, dz' \quad . \quad (2.13)$$

In the sequel we will need the following lemma

Lemma 2.2 Under the assumptions R of the problem (2.1), $Q(z,z',x,y)$ is piecewise continuously differentiable with respect to x, y and z except perhaps for $z = z'$. At $z = z'$, $Q(z,z',x,y)$ is continuous except perhaps for a finite number of points. Also $\frac{\partial Q}{\partial y} \leq 0$ and $\frac{\partial Q}{\partial x} \geq 0$.

Proof: The proof of all but the statement $\frac{\partial Q}{\partial x} \geq 0$ is given in references [1] and [11]. That $\frac{\partial Q}{\partial x} \geq 0$ is true follows in a manner exactly analogous to the proof given in [1] for $\frac{\partial Q}{\partial y} \leq 0$. ▮

Substituting equation (2.12) into equation (2.13), and using equation (2.10), we have

$$\phi(z,x,y) = g(z) + \int_y^x Q(z,z',x,y)$$

$$\times \left\{ \int_{-\infty}^0 f(s')\, e^{a(s')(z'-x)}\, ds' + \int_0^\infty h(s')\, e^{a(s')(y-z')}\, ds' \right\} dz' \quad .$$

Since R7 implies f(s) and h(s) belong to L_1 on their respective intervals, arguments similar to those used earlier allow us to change the order of integration and, after some further rearrangement, equation (2.13) becomes

$$\phi(z,x,y) = \int_{-\infty}^0 f(s')\, ds' \left\{ e^{a(s')(z-x)} + \int_y^x Q(z,z',x,y)\, e^{a(s')(z'-x)} dz' \right\}$$

$$+ \int_0^\infty h(s')\, ds \left\{ e^{a(s')(y-z)} + \int_y^x Q(z,z',x,y)\, e^{a(s')(y-z')} dz' \right\} . \tag{2.14}$$

Let

$$R_R(z,x,y,s) = e^{a(s)(z-x)} + \int_y^x Q(z,z',x,y)\, e^{a(s)(z'-x)}\, dz' \tag{2.15}$$

and

$$R_L(z,x,y,s) = e^{a(s)(y-z)} + \int_y^x Q(z,z',x,y)\, e^{a(s)(y-z')}\, dz' . \tag{2.16}$$

The subscripts R and L indicate to which side of the interval, [y,x], the R function is referred, x appearing in the exponentials in the definition of R_R and y appearing in the exponentials in the definition of R_L. They may also be thought of as referring to the right and left sides of the imaginary slab in the "pseudo-transport" problem (2.1).

Using (2.15) and (2.16) we can express the solution of the integral equation (2.12) as the sum of two integrals

$$\phi(z,x,y) = \int_{-\infty}^{0} f(s') \, R_R(z,x,y,s') \, ds' + \int_{0}^{\infty} h(s') \, R_L(z,x,y,s') \, ds' \ . \quad (2.17)$$

From this we see that $R_R(z,x,y,s)$ and $R_L(z,x,y,s)$ together form a Green's function for the problem. Because f(s) is the boundary value of N(z,s) on the right side and h(s) is the value of N(z,s) on the left, we can conclude that $R_R(z,x,y,s)$ gives the result at z due to a δ-function input from the right and $R_L(z,x,y,s)$ the result of a δ-function input from the left side of the slab. If $R_R(z,x,y,s)$ and $R_L(z,x,y,s)$ are known, the expression (2.17) implies that we can obtain the solution to the integral equations (2.12) by quadratures. Furthermore, if either f(s) or h(s) is identically zero, only one of the two R functions is needed. The result toward which we are moving is an initial value problem for these R functions.

Proceeding with the derivation we differentiate equations (2.1) with respect to x to obtain

$$\text{sgn}(s) \frac{\partial}{\partial z} N_2(z,x,y,s) + a(s)\, N_2(z,x,y,s) = k(s)\, \gamma(z) \int_{-\infty}^{\infty} N_2(z,x,y,s)\, ds' , \tag{2.18a}$$

$$N_2(y,x,y,s) = 0 \quad , \qquad s > 0 \quad , \tag{2.18b}$$

$$N_2(x,x,y,s) = -N_1(x,x,y,s) \quad , \qquad s < 0 \quad . \tag{2.18c}$$

That this differentiation is allowed is easily seen from the expression for N(z,s), (2.11), given in Theorem 2.1. The problem defined by equation (2.18) is of the same type as (2.1) with h(s) = 0. Hence, it has an equivalent integral equation, the solution to which can be written as in equation (2.17); that is

$$\psi(z,x,y) = \int_{-\infty}^{0} -N_1(x,x,y,s')\, R_R(z,x,y,s')\, ds' \quad . \tag{2.19}$$

But, as in equation (2.3), the solution to the integral equation can be written in terms of an integral of the solution of the "pseudo-transport" problem. So

$$\psi(z,x,y) = \int_{-\infty}^{\infty} N_2(z,x,y,s')\, ds' \quad . \tag{2.20}$$

Since

$$\phi(z,x,y) = \int_{-\infty}^{\infty} N(z,x,y,s')\, ds' \quad ,$$

$$\phi_2(z,x,y) = \int_{-\infty}^{\infty} N_2(z,x,y,s')\, ds' \quad . \tag{2.21}$$

Comparison of equations (2.20) and (2.21) shows that

$$\psi(z,x,y) = \phi_2(z,x,y). \tag{2.22}$$

Equation (2.18c) gives $N_2(x,x,y,s)$ in terms of $N_1(x,x,y,s)$ and the latter can be obtained directly from equation (2.1a). Thus from (2.19), (2.22) and (2.1a) we have

$$\phi_2(z,x,y) = \int_{-\infty}^{0} R_R(z,x,y,s')\Big[-a(s')\ N(x,x,y,s') + k(s')\ \gamma(x)\ \phi(x,x,z)\Big]\, ds' \tag{2.23}$$

Differentiation of equation (2.17) with respect to x supplies a second expression for $\phi_2(z,x,y)$, namely,

$$\phi_2(z,x,y) = \int_{-\infty}^{0} f(s')\ \Big[R_R(z,x,y,s')\Big]_2\ ds' + \int_{0}^{\infty} h(s')\ \Big[R_L(z,x,y,s')\Big]_2\ ds' \quad . \tag{2.24}$$

That this differentiation is allowed follows from Lemma 2.2 and the definitions of the R functions. Equating the two expressions for $\phi_2(z,x,y)$, we arrive at

$$\int_{-\infty}^{0} R_R(z,x,y,s')\ \Big[-a(s')\ N(x,x,y,s') + k(s')\ \gamma(x)\ \ \phi(x,x,y)\Big]\ ds'$$

$$- \int_{-\infty}^{0} f(s)\ \Big[R_R(z,x,y,s')\Big]_2 ds' - \int_{0}^{\infty} h(s')\ \Big[R_L(z,x,y,s')\Big]_2\ \ ds' = 0\ . \tag{2.25}$$

We can eliminate $\phi(x,x,y)$ from this expression by using equation (2.17) and $N(x,x,y,s) = f(s)$, so we finally have a relation free of ϕ and N

$$\int_{-\infty}^{0} R_R(z,x,y,s') \left\{ -a(s')\, f(s') + k(s')\, \gamma(x) \int_{-\infty}^{0} R_R(x,x,y,s'')\, f(s'')\, ds'' \right.$$

$$\left. + k(s')\, \gamma(x) \int_{0}^{\infty} R_L(x,x,y,s'')\, h(s'')\, ds'' \right\} ds'$$

$$- \int_{-\infty}^{0} f(s') \frac{\partial}{\partial x} R_R(z,x,y,s')\, ds' - \int_{0}^{\infty} h(s') \frac{\partial}{\partial x} R_L(z,x,y,s')\, ds' = 0 \; .$$

(2.26)

Since the R functions are bounded and f(s) and h(s) belong to L_1, we can change the order of integration in this equation to get

$$\int_{-\infty}^{0} f(s') \left\{ -a(s')\, R_R(z,x,y,s') - \frac{\partial}{\partial x} R_R(z,x,y,s') \right.$$

$$\left. + \gamma(x)\, R_R(x,x,y,s') \int_{-\infty}^{0} k(s'')\, R_R(z,x,y,s'')\, ds'' \right\} ds'$$

$$+ \int_{0}^{\infty} h(s') \left\{ \gamma(x)\, R_L(x,x,y,s') \int_{-\infty}^{0} k(s'')\, R_R(z,x,y,s'')\, ds'' \right.$$

$$\left. - \frac{\partial}{\partial x} R_L(z,x,y,s') \right\} ds' = 0 \quad . \qquad (2.27)$$

Now f(s) and h(s) are arbitrary functions in the class of functions satisfying the restrictions R7 and R8; hence, their coefficients must be zero, except perhaps at a finite number of points. This gives the first two imbedding equations,

$$\frac{\partial}{\partial x} R_R(z,x,y,s) = -a(s)\, R_R(z,x,y,s) + \gamma(x)\, R_R(x,x,y,s) \int_{-\infty}^{0} k(s')\, R_R(z,x,y,s')\, ds' \qquad (2.28)$$

and

$$\frac{\partial}{\partial x} R_L(z,x,y,s) = \gamma(x)\, R_L(x,x,y,s) \int_{-\infty}^{0} k(s')\, R_R(z,s,y,s')\, ds' \quad . \qquad (2.29)$$

To get the initial conditions for these equations let $y = x = \xi$ where ξ is arbitrary in $[Y,X]$. Since $y \leq z \leq x$, this implies $y = z = x$. For this case the defining equations (2.15) and (2.16) give

$$R_R(\xi,\xi,\xi,s) = 1 \quad , \qquad s < 0,\ \xi \epsilon\ [Y,X] \quad , \qquad (2.30)$$

$$R_L(\xi,\xi,\xi,s) = 1 \quad , \qquad s > 0,\ \xi \epsilon\ [Y,X] \quad . \qquad (2.31)$$

To get the other pair of imbedding equations we differentiate equation (2.1) with respect to y. Then, proceeding as above, we ultimately arrive at the relations

$$\frac{\partial}{\partial y} R_L(z,x,y,s) = a(s)\, R_L(z,x,y,s) - \gamma(y)\, R_L(y,x,y,s) \int_{0}^{\infty} k(s')\, R_L(z,x,y,s')\, ds' \quad , \qquad (2.32)$$

and

$$\frac{\partial}{\partial y} R_R(z,x,y,s) = -\gamma(y)\ R_R(y,x,y,s) \int_0^{\infty} k(s')\ R_L(z,x,y,s')\ ds' \quad , \qquad (2.33)$$

with the initial conditions (2.30) and (2.31).

Equations (2.28), (2.29), (2.32) and (2.33), along with the initial conditions (2.30) and (2.31), are the imbedding equations derived by Wing, [11]. The above derivations make clear that there is only one resolvent kernel associated with these equations. This fact will be useful later.

The solution of the imbedding equations is complicated by the fact that the dependent variables appear not only in the usual way as functions of z,x,y and s, but also with the variable z evaluated at the end points of the interval [y,x]. If the values of the R functions at the special points (x,x,y,s) and (y,x,y,s) are known, then the imbedding equations can be integrated. We can get equations for the R functions at these special points by setting z = x in equations (2.32) and (2.33) and z = y in equations (2.28) and (2.29). This results in the set

$$\frac{\partial}{\partial x} R_R(y,x,y,s) = -a(s)\ R_R(y,x,y,s)$$

$$+ \gamma(x)\ R_R(x,x,y,s) \int_{-\infty}^{0} k(s')\ R_R(y,x,y,s')\ ds' \quad , \qquad (2.34a)$$

$$\frac{\partial}{\partial y} R_R(x,x,y,s) = -\gamma(y)\ R_R(y,x,y,s) \int_0^{\infty} k(s')\ R_L(x,x,y,s')\ ds' \ , \qquad (2.34b)$$

$$\frac{\partial}{\partial x} R_L(y,x,y,s) = \gamma(x)\ R_R(x,x,y,s) \int_{-\infty}^{0} k(s')\ R_R(y,x,y,s')\ ds' \quad , \qquad (2.34c)$$

and

$$\frac{\partial}{\partial y} R_L(x,x,y,s) = a(s)\ R_L(x,x,y,s)$$

$$-\gamma(y)\ R_L(y,x,y,s) \int_{0}^{\infty} k(s')\ R_L(x,x,y,s')\ ds' \quad . \qquad (2.34d)$$

The initial conditions (2.30) and (2.31) still apply. This set of equations is analogous to Chandrasekhar's X and Y equations [4] which are of great importance in transport theory. In the sequel we refer to this set as the specialized imbedding equations. If the specialized imbedding equations can be solved, then their solutions can be inserted in the imbedding equations and these integrated. The R functions so obtained can then be used in equation (2.17) to affect the solution of the integral equation (1.1).

Both the imbedding equations and the specialized imbedding equations are integro partial differential equations and the solution of equations of this type can be difficult and cumbersome. In the next section we extend a method suggested by Allen [2] which will result in partial differential equations for the integrals appearing in the specialized imbedding equations.

III. THE METHOD OF MOMENTS

R. C. Allen [2] proposed a numerical method for the solution of a particular case of the specialized imbedding equations. The case presented was

$$\frac{\partial}{\partial t} X(t,s) = 2\gamma(t)\ Y(t,s) \int_0^\infty k(s')\ Y(t,s')\ ds' \quad ,$$

$$\frac{\partial}{\partial t} Y(t,s) = -2a(s)\ Y(t,s) + 2\gamma(t)\ X(t,s) \int_0^\infty k(s')\ Y(t,s')\ ds' \quad ,$$

$$X(0,s) = Y(0,s) = 1 \quad , \quad 0 \le t \le T \quad , \quad 0 \le s < \infty \quad .$$

These equations result when the specialized imbedding equations are restricted to the case

$$\gamma(z) = \gamma(-z) \quad ,$$

$$y = -x \quad .$$

Allen defined moments, $P_i(t)$ and $Q_i(t)$, by

$$P_i(t) = \int_0^\infty a^i(s')\ k(s')\ X(t,s')\ ds' \quad , \quad i = 0,1,2...$$

and

$$Q_i(t) = \int_0^\infty a^i(s')\ k(s')\ Y(t,s')\ ds' \quad , \quad i = 0,1,2...$$

and obtained a doubly infinite set of ordinary differential equations satisfied by them. Thus, his problem now had the form

$$\frac{\partial}{\partial t} X = 2\gamma Y Q_0 \quad ,$$

$$\frac{\partial}{\partial t} Y = - 2a\, Y + 2\gamma X Q_0 \quad ,$$

$$X(0,s) = Y(0,s) = 1 \quad ,$$

$$\frac{d}{dt} P_i = 2 \quad Q_i Q_0 \quad ,$$

$$\frac{d}{dt} Q_i = -2 \; Q_{i+1} + 2 \quad P_i Q_0 \quad ,$$

$$P_i(0) = Q_i(0) = \int_0^\infty a^i(s') \; k(s') \; ds'; i = 0,1,\ldots \quad .$$

This set was then truncated by setting

$$P_i(t) = Q_i(t) = 0 \quad , \quad i = n,n + 1,\ldots$$

and solved. Allen showed that this method required much less computational effort than the usual method of approximating the integrals by quadrature formulas and integrating the resulting set of differential equations. The prospect of substantial computational savings in itself gives sufficient motivation for attempting to apply Allen's method to the general specialized imbedding and imbedding equations.

In the remainder of this section we apply an analogous method of moments to the specialized imbedding equations. We obtain a quadruply infinite set of partial differential equations in x and y involving four moment sets. We then apply the method to the imbedding equations. Here, there also results a quadruply infinite set of partial differential equations in x and y, but now involving only two sets of moments.

Before proceeding, we make some observations which allow us to simplify our notation. Both a(s) and k(s) are assumed to be even functions of s. Since the resolvent kernel is not a function of s, we easily see from the definition of the R functions that they are also even functions of s. This means that we can write

$$\int_{-\infty}^{0} a^i(s') \, k(s') \, R_R(z,x,y,s') \, ds = \int_{0}^{\infty} a^i(s') \, k(s') \, R_R(z,x,y,s') \, ds$$

That is, we can exchange the limits of integration on any integral over s whose integrand involves only a product of a(s), k(s) and one (or both) of the R functions without affecting the result. In the following we will treat all such integrals as being over the interval $[0,\infty)$.

We now define the following moments:

$$A_i(x,y) = \int_0^{\infty} a^i(s') \, k(s') \, R_R(y,x,y,s') \, ds' \quad , \tag{3.1}$$

$$B_i(x,y) = \int_0^{\infty} a^i(s') \, k(s') \, R_L(x,x,y,s') \, ds' \quad , \tag{3.2}$$

$$C_i(x,y) = \int_0^{\infty} a^i(s') \, k(s') \, R_R(x,x,y,s') \, ds' \quad , \tag{3.3}$$

$$D_i(x,y) = \int_0^{\infty} a^i(s') \, k(s') \, R_L(y,x,y,s') \, ds' \quad , \quad i = 0,1,2,\ldots \tag{3.4}$$

At present we do not know if these integrals exist. We demonstrate their existence in Section V. To obtain partial differential equations satisfied by these moments, we multiply the specialized imbedding equations, (2.34), by $a^i(s)\ k(s)$ and integrate on s. When we interchange the order of the integration and differentiation and use the moment definitions given above we find the following set of equations for the moments:

$$\frac{\partial}{\partial x} A_i(x,y) = -A_{i+1}(x,y) + \gamma(x)\ C_i(x,y)\ A_0(x,y) \quad , \tag{3.5a}$$

$$\frac{\partial}{\partial y} C_i(x,y) = -\gamma(y)\ A_i(x,y)\ B_0(x,y) \quad , \tag{3.5b}$$

$$\frac{\partial}{\partial x} D_i(x,y) = \gamma(x) B_i(x,y)\ A_0(x,y) \quad , \tag{3.5c}$$

$$\frac{\partial}{\partial y} B_i(x,y) = B_{i+1}(x,y) - \gamma(y)\ D_i(x,y)\ B_0(x,y) \quad , i = 0,1,2,\ldots \quad . \tag{3.5d}$$

Using the initial conditions for the specialized imbedding equations

$$R_R(\xi,\xi,\xi,s) = R_L(\xi,\xi,\xi,s) = 1 \quad ,$$

we obtain the initial conditions

$$A_i(\xi,\xi) = B_i(\xi,\xi) = C_i(\xi,\xi) = D_i(\xi,\xi) = \int_0^\infty a^i(s')\ k(s')\ ds' \quad , \tag{3.5e}$$

where ξ is an arbitrary point in [Y,X]. For convenience of reference we call the set of equations, (3.5), the specialized imbedding moment equations.

To get a set of moment equations from the imbedding equations we define the moments

$$G_i(z,x,y) = \int_0^{\infty} a^i(s') \, k(s') \, R_R(z,x,y,s') \, ds' \quad , \; i = 0,1,2,\ldots \tag{3.6}$$

and

$$H_i(z,x,y) = \int_0^{\infty} a^i(s') \, k(s') \, R_L(z,x,y,s') \, ds' \quad , \; i = 0,1,2,\ldots \; . \tag{3.7}$$

The existence of these integrals is also proved in Section V. When we apply the same procedure to the imbedding equations as we did to the special imbedding equations, we obtain the set:

$$\frac{\partial}{\partial x} G_i(z,x,y) = -G_{i+1}(z,x,y) + \gamma(x) \, C_i(x,y) \, G_0(z,x,y) \quad , \tag{3.8a}$$

$$\frac{\partial}{\partial y} G_i(z,x,y) = -\gamma(y) \, A_i(x,y) \, H_0(z,x,y) \quad , \tag{3.8b}$$

$$\frac{\partial}{\partial x} H_i(z,x,y) = \gamma(x) \, B_i(x,y) \, G_0(z,x,y) \quad , \tag{3.8c}$$

$$\frac{\partial}{\partial y} H_i(z,x,y) = H_{i+1}(z,x,y) - \gamma(y) \, D_i(x,y) \, H_0(z,x,y) \; , \; i = 0,1,2,\ldots \tag{3.8d}$$

For this set of equations we can find three sets of initial conditions. We get the first set as before by using

$$R_R(\xi,\xi,\xi,s) = R_L(\xi,\xi,\xi,s) = 1$$

to obtain

$$G_i(\xi,\xi,\xi) = H_i(\xi,\xi,\xi) = \int_0^{\infty} a^i(s') \, k(s') \, ds' \quad , \; \xi \varepsilon \, [Y,X] \quad . \tag{3.8e}$$

This is not particularly useful since as we integrate away from say

(y_0, y_0, y_0), we will get only $G_i(y_0, x, y_0)$ and so on. The second and third sets are obtained when we note that the moments $G_i(z,x,y)$ and $H_i(z,x,y)$ can be related to the moments $A_i(x,y)$, $B_i(x,y)$, $C_i(x,y)$ and $D_i(x,y)$. Specifically,

$$G_i(\xi,\xi,y) = C_i(\xi,y) \quad , \quad Y \le y \le \xi \le X \quad , \tag{3.8f}$$

$$G_i(\xi,x,\xi) = A_i(x,\xi) \quad , \quad Y \le \xi \le x \le X \quad , \tag{3.8g}$$

$$H_i(\xi,\xi,y) = B_i(\xi,y) \quad , \quad Y \le y \le \xi \le X \quad , \tag{3.8h}$$

$$H_i(\xi,x,\xi) = D_i(x,\xi) \quad , \quad Y \le \xi \le x \le X \quad , \quad i = 0,1,2,\ldots \; . \tag{3.8i}$$

The above quadruply infinite set of equations, (3.8a-d), with any consistent set of the initial conditions, we call the imbedding moment equations.

For ease of reference we collect all these equation sets, using the moment notation, in the order in which their solution should be carried out.

1. Specialized imbedding moment equations.

$$\frac{\partial}{\partial x} A_i(x,y) = -A_{i+1}(x,y) + \gamma(x)\, C_i(x,y)\, A_0(x,y) \quad , \tag{3.9a}$$

$$\frac{\partial}{\partial y} C_i(x,y) = -\gamma(y)\, A_i(x,y)\, B_0(x,y) \quad , \tag{3.9b}$$

$$\frac{\partial}{\partial x} D_i(x,y) = \gamma(x)\, B_i(x,y)\, A_0(x,y) \quad , \tag{3.9c}$$

$$\frac{\partial}{\partial y} B_i(x,y) = B_{i+1}(x,y) - \gamma(y)\, D_i(x,y)\, B_0(x,y) \quad , \tag{3.9d}$$

$$A_i(\xi,\xi) = B_i(\xi,\xi) = C_i(\xi,\xi) = D_i(\xi,\xi) = \int_0^\infty a^i(s')\, k(s')\cdot ds' \quad ,$$

$$i = 0,1,2,\ldots \; . \tag{3.9e}$$

2. Specialized imbedding equations.

$$\frac{\partial}{\partial x} R_R(y,x,y,s) = -a(s)\ R_R(y,x,y,s) + \gamma(x)\ R_R(x,x,y,s)\ A_0(x,y) \quad , \qquad (3.10a)$$

$$\frac{\partial}{\partial y} R_R(x,x,y,s) = -\gamma(y)\ R_R(y,x,y,s)\ B_0(x,y) \quad , \qquad (3.10b)$$

$$\frac{\partial}{\partial x} R_L(y,x,y,s) = \gamma(x)\ R_L(x,x,y,s)\ A_0(x,y) \quad , \qquad (3.10c)$$

$$\frac{\partial}{\partial y} R_L(x,x,y,s) = a(s)\ R_L(x,x,y,s) - \gamma(y)\ R_L(y,x,y,s)\ B_0(x,y) \quad , \qquad (3.10d)$$

$$R_R(\xi,\xi,\xi,s) = R_L(\xi,\xi,\xi,s) = 1 \quad , \quad Y \le \xi \le X \quad . \qquad (3.10e)$$

The moments obtained from the solution of the specialized imbedding moment equations are used to solve the

3. Imbedding moment equations.

$$\frac{\partial}{\partial x} G_i(z,x,y) = -G_{i+1}(z,x,y) + \gamma(x)\ C_i(x,y)\ G_0(z,x,y) \quad , \qquad (3.11a)$$

$$\frac{\partial}{\partial y} G_i(z,x,y) = -\gamma(y)\ A_i(x,y)\ H_0(z,x,y) \quad , \qquad (3.11b)$$

$$\frac{\partial}{\partial x} H_i(z,x,y) = \gamma(x)\ B_i(x,y)\ G_0(z,x,y) \quad , \qquad (3.11c)$$

$$\frac{\partial}{\partial y} H_i(z,x,y) = H_{i+1}(z,x,y) - \gamma(y)\ D_i(x,y)\ H_0(z,x,y) \quad , \qquad (3.11d)$$

with

$$G_i(\xi,\xi,\xi) = H_i(\xi,\xi,\xi) = \int_0^\infty a^i(s')\ k(s')\ ds' \quad , \quad \xi \varepsilon\ [Y,X] \quad ,$$

$$i = 0,1,2,\ldots\ ; \qquad (3.11e)$$

or

$$G_i(\xi,\xi,y) = C_i(\xi,y) \quad , \quad Y \le y \le \xi \le X \quad , \tag{3.11f}$$

$$H_i(\xi,\xi,y) = B_i(\xi,y) \quad , \quad Y \le y \le \xi \le X \quad , \tag{3.11g}$$

or

$$G_i(\xi,x,\xi) = A_i(x,\xi) \quad , \quad Y \le \xi \le x \le X \quad , \tag{3.11h}$$

$$H_i(\xi,x,\xi) = D_i(x,\xi) \quad , \quad Y \le \xi \le x \le X \quad , \quad i = 0,1,2,\ldots \; . \tag{3.11i}$$

$G_0(z,x,y)$ and $H_0(z,x,y)$ from the solution of the imbedding moment equations, along with $R_R(x,x,y,s)$, $R_R(y,x,y,s)$, $R_L(x,x,y,s)$ and $R_L(y,x,y,s)$ from the solution of the specialized imbedding equations, are then used to solve the

4. Imbedding equations

$$\frac{\partial}{\partial x} R_R(z,x,y,s) = -a(s)\, R_R(z,x,y,s) + \gamma(x)\, R_R(x,x,y,s)\, G_0(z,x,y) \quad , \tag{3.12a}$$

$$\frac{\partial}{\partial y} R_R(z,x,y,s) = -\gamma(y)\, R_R(y,x,y,s)\, H_0(z,x,y) \quad , \tag{3.12b}$$

$$\frac{\partial}{\partial x} R_L(z,x,y,s) = \gamma(x)\, R_L(x,x,y,s)\, G_0(z,x,y) \quad , \tag{3.12c}$$

$$\frac{\partial}{\partial y} R_L(z,x,y,s) = a(s)\, R_L(z,x,y,s) - \gamma(y)\, R_L(y,x,y,s)\, H_0(z,x,y) \quad , \tag{3.12d}$$

with

$$\left\{ \begin{array}{l} R_R(\xi,\xi,\xi,s) = 1 \quad , \quad s < 0 \quad , \quad \xi \varepsilon [Y,X] \quad , \qquad (3.12e) \\ R_L(\xi,\xi,\xi,s) = 1 \quad , \quad s > 0 \quad , \quad \xi \varepsilon [Y,X] \quad , \qquad (3.12f) \end{array} \right.$$

or

$$R_R(\xi,\xi,y,s) = R_R(x,x,y,s) \quad , \quad \text{at } \xi = x \quad , \tag{3.12g}$$

$$R_L(\xi,\xi,y,s) = R_L(x,x,y,s) \quad , \quad \text{at } \xi = x \quad , \tag{3.12h}$$

or

$$R_R(\xi,x,\xi,s) = R_R(y,x,y,s) \quad , \quad \text{at } \xi = y \quad , \tag{3.12i}$$

$$R_L(\xi,x,\xi,s) = R_L(y,x,y,s) \quad , \quad \text{at } \xi = y \quad . \tag{3.12j}$$

In equations (3.12g) through (3.12j) the notation is intended to indicate that the solutions of the imbedding equations can be started from the solutions of the specialized imbedding equations.

It is not always necessary to solve all the equations appearing above. For instance, if $h(s) \equiv 0$, we need only find $R_R(z,x,y,s)$. This means that only one of the equations (3.12) must be integrated. Choosing equation (3.12a) we then require $R_R(x,x,y,s)$ and $G_0(z,x,y)$. $G_0(z,x,y)$ can be obtained from the integration of the single set (3.11a) while $R_R(x,x,y,s)$ is secured by solving the pair of equations (3.10a), (3.10b). To integrate the latter pair requires a knowledge of $C_i(x,y)$, $A_0(x,y)$ and $B_0(x,y)$. We will show later that it is always the case that $A_0(x,y) = B_0(x,y)$; hence, we can get all three functions by solving only the sets (3.9a) and (3.9b). Even if both $f(s)$ and $h(s)$ are non zero, some reduction in the imbedding and imbedding moment equations is possible.

At this time we do not discuss the existence and uniqueness of solutions to these sets of equations, nor do we consider the convergence of the solutions of the truncated sets of imbedding moment and specialized imbedding moment equations to the solutions of the infinite sets. In

Section VI we show that for the case $\gamma(z)$ = constant, there is always some region in which there exists a unique solution to which the solutions of the truncated equations converge. The existence, uniqueness and convergence has not been proven for the case of non constant $\gamma(z)$.

In the next section we take a closer look at the moments.

IV. A PROPERTY OF THE MOMENTS

We will discover that the moments defined in the preceding section are intimately connected to the resolvent kernel of the integral equation (1.1). In the following certain properties of this resolvent kernel are required. The first of these is the exchange relation

$$\gamma(z)\, Q(z,z',x,y) = \gamma(z')\, Q(z',z,x,y) \quad . \tag{4.1}$$

To establish this relation it is sufficient to show that the product series of $\gamma(z)$ or $\gamma(z')$ and the appropriate Neumann series expansion for the kernel are equal term by term. That is

$$\gamma(z)\, K_n(z,z') = \gamma(z')\, K_n(z',z) \tag{4.2}$$

where $K_n(r,t)$ is the nth iterated kernel of the kernel defined in equation (1.1b); it is given by

$$K_n(r,t) = \int_y^x \dots (n-1) \dots \int_y^x \gamma(t_1)\dots\gamma(t_{n-1})\, \gamma(t)\, dt_1 \dots dt_{n-1}$$

$$\times \int_0^\infty k(s_1)\, e^{-a(s_1)|r-t_1|} ds_1 \dots \int_0^\infty k(s_n)\, e^{-a(s_n)|t_{n-1}-t|} ds_n \quad . \tag{4.3}$$

Thus the relation (4.2) is merely the identity

$$\gamma(z)\int_y^x \ldots(n-1)\ldots \int_y^x \gamma(t_1)\ldots\gamma(t_{n-1})\ \gamma(z')\ dt_1\ldots dt_{n-1}$$

$$\times\int_0^\infty k(s_1)\ e^{-a(s_1)|z-t_1|}\ ds_1\ldots \int_0^\infty k(s_n)\ e^{-a(s_n)|t_{n-1}-z'|}\ ds_n$$

$$= \gamma(z')\int_y^x \ldots(n-1)\ldots \int_y^x \gamma(t_1)\ldots\gamma(t_{n-1})\ \gamma(z)\ dt_1\ldots dt_{n-1}$$

$$\times\int_0^\infty k(s_1)\ e^{-a(s_1)|z'-t_1|}\ ds_1\ldots \int_0^\infty k(s_n)\ e^{-a(s_n)|t_n-z|}\ ds_n \quad . \tag{4.4}$$

The other relations we need are the Fredholm identities. It is well known that the resolvent kernel of a Fredholm integral equation satisfies integral equations involving the kernel of the original equation [8], [10]. These integral equations are the Fredholm identities:

$$Q(z,z',x,y) = K(z,z') + \int_y^x K(z,t)\ Q(t,z',x,y)\ dt \quad , \tag{4.5}$$

$$Q(z,z',x,y) = K(z,z') + \int_y^x K(t,z')\ Q(z,t,x,y)\ dt \quad . \tag{4.6}$$

For our kernel these become

$$Q(z,z',x,y) = \gamma(z')\int_0^\infty k(s')\, e^{-a(s')|z-z'|}\, ds'$$

$$+ \int_y^x \gamma(t)\, Q(t,z',x,y)\, dt \int_0^\infty k(s')\, e^{-a(s')|t-z|}\, ds' \;,$$

(4.7)

$$Q(z,z',x,y) = \gamma(z')\int_0^\infty k(s')\, e^{-a(s')|z-z'|}\, ds'$$

$$+ \gamma(z')\int_y^x Q(z,t,x,y)\, dt \int_0^\infty k(s')\, e^{-a(s')|t-z'|}\, ds'$$

(4.8)

Using these relations we now show that, in general,

$$A_0(x,y) = B_0(x,y) \quad .$$

We first establish the Lemma:

Lemma 4.1. The R functions and the resolvent kernel satisfy

$$\int_0^\infty k(s')\, R_R(z,x,y,s')\, ds' = Q(z,x,x,y)/\gamma(x) \quad , \tag{4.9}$$

$$\int_0^\infty k(s')\, R_L(z,x,y,s')\, ds' = Q(z,y,x,y)/\gamma(y) \quad . \tag{4.10}$$

Proof: We multiply the defining equation of $R_R(z,x,y,s\ )$, equation (2.15), by $k(s)$ and integrate on s. This yields

$$\int_0^\infty k(s') \, R_R(z,x,y,s') \, ds' = \int_0^\infty k(s') \, e^{a(s')(z-x)} \, ds'$$

$$+ \int_0^\infty k(s') \, ds' \int_y^x Q(z,z',x,y) \, e^{a(s')(z'-x)} \, dz'$$

$$= \int_0^\infty k(s') \, e^{a(s')(z-x)} \, ds'$$

$$+ \int_y^x Q(z,z',x,y) \, dz' \int_0^\infty k(s') \, e^{a(s')(z'-x)} \, ds'$$

$$= \frac{1}{\gamma(x)} \left\{ \gamma(x) \int_0^\infty k(s') \, e^{a(s')(z-x)} \, ds' \right.$$

$$\left. + \gamma(x) \int_y^x Q(z,z',x,y) \, dz' \int_0^\infty k(s') \, e^{a(s')(z'-x)} \, ds' \right\} \quad ,$$

which, by equation (4.8), is equal to $Q(z,x,x,y)/\gamma(x)$. Relation (4.10) is proved in an analogous manner. ▮

Theorem 4.2 Under the restrictions required to establish Theorem 2.1,

$$A_0(x,y) = B_0(x,y) \quad . \tag{4.11}$$

Proof: From the definition of $A_i(x,y)$, equation (3.1), we have

$$A_0(x,y) = \int_0^\infty k(s') \, R_R(y,x,y,s') \, ds' \quad .$$

Using Lemma 4.1 and the exchange relation (4.1), we see that

$$A_0(x,y) = Q(x,y,x,y)/\gamma(y) \quad .$$

Now from the definition of $B_i(x,y)$, equation (3.2), we see that

$$B_0(x,y) = \int_0^\infty k(s') \, R_L(x,x,y,x') \, ds' \quad ,$$

and by Lemma 4.1,

$$B_0(x,y) = Q(x,y,x,y)/\gamma(y) \quad .$$

Equating, we have the result. ▮

Theorem 4.2 is of essential importance in the later development. In Appendix A we present a continuation of this section which is of interest because it displays the relationship of the moments to the resolvent kernel.

V. POWER SERIES REPRESENTATION OF THE KERNEL, K(z,z´)

In this section we prove some results about a power series representation of the kernel, K(z,z´), given by equation (1.1b). These results are used later in the proof of the existence and uniqueness of solutions to the imbedding and imbedding moment equations. They will also show that the Taylor's series expansions for the resolvent kernel derived in Appendix A have a non-zero radius of convergence.

Theorem 5.1. For all admissible functions, k(s), a(s) and γ(z), (that is, functions satisfying restrictions R1, R2, R3, R4 and R6), the kernel of the integral equation (1.1) can be expressed in the form

$$K(z,z') = \gamma(z') \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} |z - z'|^j \int_0^{\infty} k(s') \, a^j(s') \, ds' \tag{5.1}$$

where the series is convergent for all values of z and z´ belonging to the interval of definition of K(z,z´), [Y,X]. Moreover, the series is absolutely and uniformly convergent for all $z,z' \in [Y,X]$.

Proof: The proof proceeds in two steps. We first show that the change in order of integration and summation can be carried out when the domain of k(s) is finite. Assume that this domain is $[0,s_0] < \infty$. Then, since a(s) is piecewise continuous on the bounded interval $[0,s_0]$, a(s) is bounded. Let

$$|a(s)| \leq A_{max} < \infty \, , \; 0 \leq s \leq s_0 \quad .$$

Now consider the infinite series

$$\sum_{j=0}^{\infty} \int_0^{s_0} \frac{|z - z'|^j}{j!} |k(s')| \; |a^j(s')| \; ds'$$

$$= \sum_{j=0}^{\infty} \frac{|z - z'|^j}{j!} \int_0^{s_0} |k(s')| \; |a^j(s')| \; ds'$$

This series converges for all z and z´ in [Y,X] because

$$\sum_{j=0}^{\infty} \frac{|z - z'|^j}{j!} \int_0^{s_0} |k(s')| \; |a^j(s')| \; ds'$$

$$\leq \sum_{j=0}^{\infty} \frac{|z - z'|^j}{j!} A^j_{max} \int_0^{s_0} |k(s')| \; ds'$$

$$\leq \sum_{j=0}^{\infty} \frac{|z - z'|^j}{j!} A^j_{max} ||k(s)||_{L_1}$$

$$\leq e^{A_{max}|z - z'|} ||k(s)||_{L_1}$$

$$\leq e^{A_{max}|X - Y|} ||k(s)||_{L_1}$$

and $k(s) \varepsilon \; L_1[0,\infty)$ from R2. Then

$$\int_0^{s_0} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} k(s') \; a^j(s') \, | \, z - z' |^j \; ds'$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} |z - z'|^j \int_0^{s_0} k(s') \; a^j(s) \; ds'$$

by the absolute convergence termwise integration Thereom [7].

If the range of k(s) is infinite while the range of a(s) is finite, the above proof is valid if the zeroth term of the series is split off. This term is merely the infinite integral of k(s) which is finite because k(s) belongs to L_1.

Now assume that the range of both a(s) and k(s) is infinite. Splitting up the summation we have

$$\int_0^\infty k(s') e^{-a(s') |z-z'|} ds'$$

$$= \int_0^\infty k(s') \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} |z - z'|^j a^j(s') \right) ds'$$

$$= \int_0^\infty k(s') \sum_{j=0}^{n} \frac{(-1)^j}{j!} a^j(s') |z - z'|^j ds'$$

$$+ \int_0^\infty k(s') \sum_{j=n+1}^{\infty} \frac{(-1)^j}{j!} a^j(s') |z - z'|^j ds'$$

Because the first series on the right is finite we can change the order of integration and summation so it follows that

$$\left| \int_0^\infty k(s') \, e^{-a(s')|z-z'|} \, ds' - \sum_{j=0}^{n} \frac{(-1)^j}{j!} |z - z'|^j \int_0^\infty k(s') \, a^j(s') \, ds' \right|$$

$$\leq \int_0^\infty |k(s')| \sum_{j=n+1}^{\infty} |a^j(s')| \frac{|z - z'|^j}{j!} \, ds'$$

By a corollary to the monotone convergence theorem the right side of this inequality can be written as

$$\sum_{j=n+1}^{\infty} \left\{ \int_0^\infty |k(s')| \; |a^j(s')| \, ds' \right\} \frac{|z - z'|^j}{j!}$$

$$\leq \sum_{j=n+1}^{\infty} \frac{j!}{(X - Y)^j} \; r_j \frac{|z - z'|^j}{j!}$$

$$\leq \sum_{j=n+1}^{\infty} r_j$$

where the condition R3 has been used. This last term can be made as small as desired since the r_j form a convergent series. Hence

$$\int_0^\infty k(s') \, e^{-a(s')|z-z'|} \, ds' = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} |z - z'|^j \int_0^\infty k(s') \, a^j(s') \, ds'$$

for any z, z', belonging to $[Y,X]$. Equation (5.1) now follows by multiplication by the bounded piecewise continuous function $\gamma(z)$. Since the integral in (1.1b) is a continuous function of z and z', $K(z,z')$ is a bounded piecewise continuous function of z and z' on the interval of definition $[Y,X]$. That the series of equation (5.1) converges follows from the above proof and the boundedness of $K(z,z')$. Since this series is a power series, it is absolutely and uniformly convergent inside its radius of convergence. As the proof shows, we are allowed to take the maximum value of $|z - z'|$ in equation (5.1) to estimate the radius of convergence and this value is $X - Y$. From the convergence of the series in (5.1) we have

$$\operatorname*{Lim\,Sup}_{j \to \infty} (X - Y) \left[\frac{\left| \int_0^\infty k(s')\, a^j(s')\, ds' \right|}{j!} \right]^{1/j} \leq 1$$

so that

$$\operatorname*{Lim\,Sup}_{j \to \infty} \left[\frac{\left| \int_0^\infty k(s') \, a^j(s') \, ds' \right|}{j!} \right]^{1/j} \leq \frac{1}{X - Y} .$$

But the quantity on the left is by definition the reciprocal of the radius of convergence so we can conclude that X - Y is less than the radius of convergence. From this the series is absolutely and uniformly convergent for all z, z' in [Y,X]. ∎

For later reference the result derived in the last part of the proof is stated in a corollary.

Corollary 5.2.

$$\operatorname*{Lim\,Sup}_{j \to \infty} \left[\frac{\left| \int_0^\infty k(s') \, a^j(s') \, ds' \right|}{j!} \right]^{1/j} \leq \frac{1}{X - Y} \qquad (5.2)$$

We also need the following two corollaries.

Corollary 5.3.

$$\operatorname*{Lim\,Sup}_{j \to \infty} \left[\frac{\left| \int_0^\infty R_w(z,x,y,s') \, k(s') \, a^j(s') \, ds' \right|}{j!} \right]^{1/j} \leq \frac{1}{X - Y} , \qquad (5.3)$$

$w = R$ or L, $y \leq z \leq x$, except perhaps at a finite number of z points.

Proof: From the definitions of the R functions, it is easily seen that they are bounded provided the integrals involving the resolvent kernel are bounded. However, the resolvent kernel is an L_1 function in both z' and z, and by Lemma 2.2 it is continuous except perhaps for a finite number of points. Thus, the integrals, and hence, the R functions, exist and are bounded except at a finite number of points. For the points where $R_w(z,x,y,s')$ is bounded we have

$$\sum_{j=0}^{\infty} \frac{|z - z'|^j}{j!} \left| \int_0^{\infty} R_w(z,x,y,s')\ k(s')\ a^j(s')\ ds' \right|$$

$$\leq M \sum_{j=0}^{\infty} \frac{|z - z'|^j}{j!} \left| \int_0^{\infty} k(s')\ a^j(s')\ ds' \right|$$

and the series on the right is absolutely and uniformly convergent with radius of convergence greater than X - Y. We conclude that the series on the left must have at least the same radius of convergence, hence, equation (5.3) follows. ▮

Corollary 5.4 The integrals in equations (3.1) through (3.4), (3.6) and (3.7), which define the moments A_i, B_i, C_i, D_i, G_i, and H_i exist.

Proof: This is an immediate consequence of Corollary 5.3. ▮

An examination of the power series developed in Appendix A shows that Corollary 5.3 is sufficient to ensure their absolute and uniform convergence except for a finite number of z points.

We now proceed to examine the existence and uniqueness of solutions to the various imbedding equations for the special case when $\gamma(z)$ is a constant.

VI. SOLUTION WITH $\gamma(z)$ CONSTANT

We now show that the imbedding moment equations possess unique solutions on some non-vanishing interval if $\gamma(z)$ is a constant. For this we use the Neumann series expansion for the resolvent kernel

$$Q(z,z',x,y) = \sum_{n=1}^{\infty} N_n(z,z',x,y) \qquad (6.1)$$

where $N_n(z,z',x,y)$ is the n-fold iterated kernel defined by

$$N_1(z,z',x,y) = K(z,z') \quad ,$$

$$\vdots$$

$$N_i(z,z',x,y) = \int_y^x N_{i-1}(z,t)\, N_1(t,z')\, dt \quad .$$

$$\vdots$$

We recall that this series is almost uniformly convergent whenever

$$||K(z,z')||_{L_2} < 1 \qquad (6.2)$$

We now derive some results concerning the R functions and the various moments for the case where $\gamma(z)$ is a constant.

<u>Theorem 6.1</u> If $\gamma(z) = c$, a constant, then for $d \geq 0$, $y_o \leq z_o \leq y_o + d$ and for $\Delta \geq 0$

$$R_R(z_o,y_o+d,y_o,s) = R_R(z_o+\Delta,y_o+\Delta+d,y_o+\Delta,s) \quad , \qquad (6.3)$$

$$R_L(z_o,y_o+d,y_o,s) = R_L(z_o+\Delta,y_o+\Delta+d,y_o+\Delta,s) \quad . \qquad (6.4)$$

Proof. We prove only equation (6.3); (6.4) is proved in a similar way.

Using the definition of $R_R(z,x,y,s)$, equation (2.15), we obtain

$$R_R(z_o,y_o+d,y_o,s) = e^{a(s)(z_o-y_o-d)}$$

$$+ \int_{y_o}^{y_o+d} Q(z_o,z',y_o+d,y_o)\ e^{a(s)(z'-y_o-d)}\ dz'$$

and

$$R_R(z_o+\Delta,y_o+\Delta+d,y_o+\Delta,s) = e^{a(s)(z_o+\Delta-y_o-\Delta-d)}$$

$$+ \int_{y_o+\Delta}^{y_o+\Delta+d} Q(z_o+\Delta,z',y_o+\Delta+d,y_o+\Delta)e^{a(s)(z'-y_o-\Delta-d)}\ dz'$$

$$= e^{a(s)(z_o-y_o-d)} + \int_{y_o+\Delta}^{y_o+\Delta+d} Q(z_o+\Delta,z',y_o+\Delta+d,y_o+\Delta)\ e^{a(s)(z'-y_o-\Delta-d)}\ dz' \quad .$$

Thus equation (6.3) will be demonstrated if we show that

$$\int_{y_o}^{y_o+d} Q(z_o,z',y_o+d,y_o)\ e^{a(s)(z'-y_o-d)}\ dz'$$

$$= \int_{y_o+\Delta}^{y_o+\Delta+d} Q(z_o+\Delta,z',y_o+\Delta+d,y_o+\Delta)\ e^{a(s)(z'-y_o-\Delta-d)}\ dz' \quad .$$

Substitute the series expression for the resolvent kernel in each of the integrals appearing above. Since almost uniform convergence allows us to integrate termwise, we can change the order of integration and summation. This means that it is sufficient to show that

$$\int_{y_o}^{y_o+d} N_n(z_o,z',y_o+d,y_o)\ e^{a(s)(z'-y_o-d)}\ dz'$$

$$= \int_{y_o+\Delta}^{y_o+\Delta+d} N_n(z_o+\Delta,z',y_o+\Delta+d,y_o+\Delta)\ e^{a(s)(z'-y_o-\Delta-d)}\ dz' \quad . \quad (*)$$

For $\gamma(z')=c$, the n-fold iterated kernel can be written as

$$N_n(z,z',x,y) = c^n \int_y^x \ldots(n-1)\ldots \int_y^x dt_1\ldots$$

$$\ldots dt_n \int_0^\infty k(s_1)\ e^{a(s_1)|z-t_1|}\ ds_1 \ldots \int_0^\infty k(s_i)\ e^{-a(s_i)|t_{i-1}-t_i|}\ ds_i$$

$$\ldots \int_0^\infty k(s_n)\ e^{-a(s_n)|t_{n-1}-t_n|}\ ds_n \quad .$$

So, letting $z'=t_n$, we can write the left hand side of equation (*) as

$$\int_{y_o}^{y_o+d} N_n(z_o,t_n,y_o+d,y_o)\, e^{a(s)(t_n-y_o-d)}\, dt_n$$

$$= c^n \int_{y_o}^{y_o+d} \ldots(n)\ldots \int_{y_o}^{y_o+d} dt_1 \ldots dt_n \int_0^\infty k(s_1)\, e^{-a(s_1)|z_o-t_1|} ds_1 \ldots$$

$$\ldots \int_0^\infty k(s_i)\, e^{-a(s_i)|t_{i-1}-t_i|} ds_i$$

$$\ldots \int_0^\infty k(s_n)\, e^{-a(s_n)|t_{n-1}-t_n|} ds_n\, e^{a(s)(t_n-y_o-d)} \quad . \qquad (**)$$

We now show that the right side of equation (*) reduces to the right side of equation (**). Let $z'=r_n$ in the right hand side of equation (*). Then

$$\int_{y_o+\Delta}^{y_o+\Delta+d} N_n(z_o+\Delta,r_n,y_o+\Delta+d,y_o+\Delta)\, e^{a(s)(r_n-y_o-\Delta-d)}\, dr_n$$

$$= c^n \int_{y_o+\Delta}^{y_o+\Delta+d} \ldots(n)\ldots \int_{y_o+\Delta}^{y_o+\Delta+d} dr_1 \ldots dr_n \int_0^\infty k(s_1)\, e^{-a(s_1)|z_o+\Delta-r_1|} ds_1$$

$$\ldots \int_0^\infty k(s_1)\, e^{-a(s_i)|t_{i-1}-t_i|} ds_i \ldots \int_0^\infty k(s_n) e^{-a(s_n)|r_{n-1}-r_n|} ds_n$$

$$\times\ e^{a(s)(r_n-y_o-\Delta-d)} \quad . \qquad (***)$$

Further let

$$t_1 = r_1 - \Delta$$
$$\vdots$$
$$t_i = r_i - \Delta$$
$$\vdots$$
$$t_n = r_n - \Delta \quad .$$

This changes the limits of integration in the following way:

$$y_o + \Delta \to y_o \text{ and } y_o + \Delta + d \to y_o + d$$

Moreover,

$$|r_{i-1} - r_i| = |r_{i-1} - \Delta - r_i + \Delta| = |t_{i-1} - t_i|$$

and

$$r_n - \Delta - y_o - d = t_n - y_o - d \quad .$$

Making these changes in the right hand side of equation (***) we finally have

$$\int_{y_o+\Delta}^{y_o+\Delta+d} N_n(z_o+\Delta, r_n, y_o+\Delta+d, y_o+\Delta)\, e^{a(s)(r_n-y_o-\Delta-d)}\, dr_n$$

$$= c^n \int_{y_o}^{y_o+d} \ldots(n)\ldots \int_{y_o}^{y_o+d} dt_1 \ldots dt_n \int_0^\infty e^{-a(s_1)|z_o-t_1|}\, ds_1 \ldots$$

$$\ldots \int_0^\infty k(s_i)\, e^{-a(s_i)|t_{i-1}-t_i|}\, ds_i \ldots$$

$$\ldots \int_0^\infty k(s_n)\, e^{-a(s_n)|t_{n-1}-t_n|}\, ds_n \; e^{a(s)(t_n-y_o-d)}$$

The right side of this equation is exactly the right side of equation (**) and the theorem is proved. ▌

Corollary 6.2. For $\gamma(z) = c$, a constant, $R_R(y,x,y,s)$, $R_R(x,x,y,s)$, $R_L(y,x,y,s)$ and $R_L(x,x,y,s)$ are constant on lines parallel to the line $x = y$. That is

$$R_R(y,x,y,s) = R_R(y+\Delta,x+\Delta,y+\Delta,s) \quad , \text{ etc.}$$

Proof: We use the diagram below to make the argument clear. Any point on a line parallel to the line $x = y$ can be given in terms of some (arbitrary) initial point (y_o,y_o), lying on the line $x = y$, in terms of equal increments of length Δ in both the x and y coordinate directions and a single increment d in the x coordinate only. This is illustrated in Fig. 1. To show that $R_R(y,x,y,s)$ is constant on the

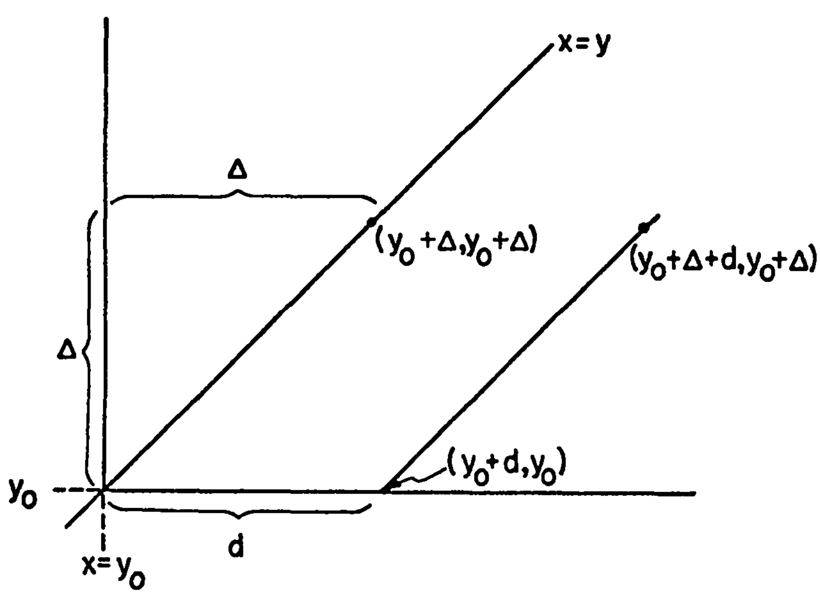

Fig. 1 A line of constant R values.

line through the points (y_o+d,y_o) and $(y_o+\Delta+d,y_o+\Delta)$ we need to show that

$$R_R(y_o,y_o+d,y_o,s) = R_R(y_o+\Delta,y_o+\Delta+d,y_o+\Delta,s) \quad .$$

But this is just equation (6.3) evaluated at $z_o = y_o$. The statement for $R_R(x,x,y,s)$ is also obtained from equations (6.3) evaluated at $z_o = y_o + d$. The statements for $R_L(y,x,y,s)$ and $R_L(x,x,y,s)$ follow from (6.4) with $z_o = y_o$ and $z_o = y_o + d$ respectively. ∎

<u>Corollary 6.3.</u> For $\gamma(z) = c$, a constant, the moments $A_i(x,y)$, $B_i(x,y)$, $C_i(x,y)$ and $D_i(x,y)$ are constant on lines parallel to the line $x = y$. That is

$$A_i(x,y) = A_i(x+\Delta,y+\Delta) \quad , \text{etc.}$$

Proof: The assertion for $A_i(x,y)$ follows immediately from the definition of A_i and Corollary 6.2 since

$$A_i(y_o+d,y_o) = \int_0^\infty k(s')\ a^i(s')\ R_R(y_o,y_o+d,y_o s')\ ds'$$

$$= \int_0^\infty k(s')a^i(s')\ R_R(y_o+\Delta+d,y_o+\Delta,s')\ ds'$$

$$= A_i(y_o+\Delta+d,y_o+\Delta) \quad .$$

The statements for the remaining moments are similarly proven from their definitions and Corollary 6.2. ∎

<u>Theorem 6.4.</u> For $\gamma(z)=c$, a constant,

$$R_R(y,x,y,s) = R_L(x,x,y,s) \quad , \tag{6.5}$$

$$R_R(x,x,y,s) = R_L(y,x,y,s) \quad . \tag{6.6}$$

Proof. We pick an arbitrary point, (x_1,y_1), in the region of interest, the region below and to the right of the line $x = y$. Assuming (x_1,x_1) and (y_1,y_1) are on the line $x = y$, this point can be reached by moving downward from the point (x_1,x_1) a distance $\Delta = x_1 - y_1$ along the line $x = x_1$ or by moving horizontally to the right from the point (y_1,y_1) the same distance Δ. See Fig. 2.

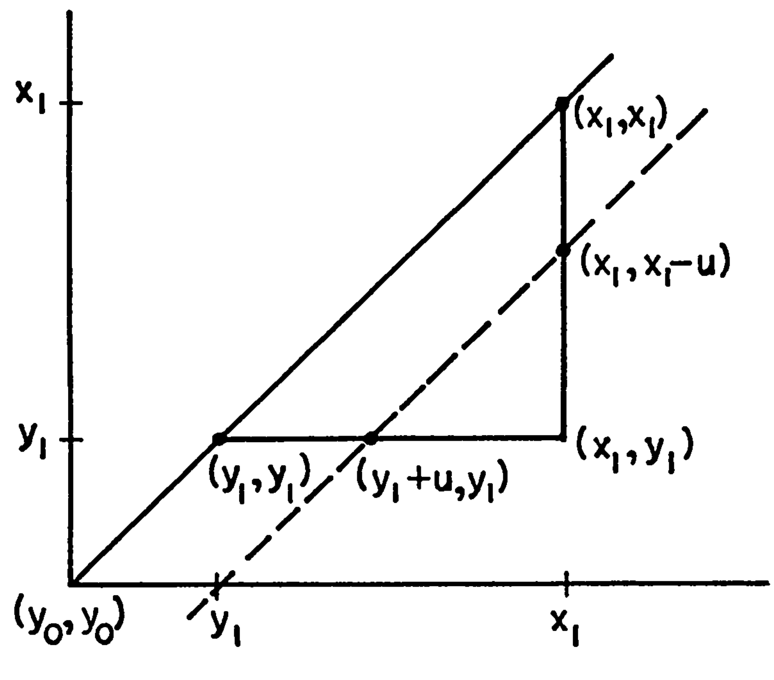

Fig. 2 Illustrative diagram for Theorem 6.4.

Now consider the specialized imbedding equations restricted to these paths

$$\frac{\partial}{\partial x} R_R(y_1,x,y_1,s) = -a(s)\ R_R(y_1,x,y_1,s) + c\ R_R(x,x,y_1,s)\ A_o(x,y_1), \tag{6.7}$$

$$\frac{\partial}{\partial y} R_R(x_1,x_1,y,s) = -c\ R_R(y,x_1,y,s)\ A_o(x_1,y) \quad , \tag{6.8}$$

$$\frac{\partial}{\partial y} R_L(x_1,x_1,y,s) = a(s)\ R_L(x_1,x_1,y,s) - c\ R_L(y,x_1,y,s)\ A_o(x,y) \quad , \tag{6.9}$$

$$\frac{\partial}{\partial x} R_L(y_1,x,y_1,s) = c\ R_L(x,x,y_1,s)\ A_o(x,y_1) \quad . \tag{6.10}$$

These equations can be considered as ordinary differential equations if the variable s is held constant. We shift to a new single variable u such that u = 0 on the line x = y; then dx = du. We set dy = -du since the equations involving the partial derivatives with respect to y need to be integrated in the negative y direction. With these changes the specialized imbedding equations above become

$$\frac{\partial}{\partial u} R_R(y_1,y_1+u,y_1,s) = -a(s)\ R_R(y_1,y_1+u,y_1,s)$$

$$+ c\ R_R(y_1+u,y_1+u,y,s)\ A_o(y_1+u,y_1) \quad , \tag{6.11a}$$

$$\frac{\partial}{\partial u} R_R(x_1,x_1,x_1-u,s) = c\ R_R(x_1-u,x_1,x_1-u,s)\ A_o(x_1,x_1-u) \quad , \tag{6.11b}$$

$$\frac{\partial}{\partial u} R_L(x_1,x_1,x_1-u,s) = -a(s)\ R_L(x_1,x_1,x_1-u,s)$$

$$+ c\ R_L(x_1-u,x_1,x_1-u,s)\ A_o(x_1,x_1-u) \quad , \tag{6.11c}$$

$$\frac{\partial}{\partial u} R_L(y_1,y_1+u,y_1,s) = c\, R_L(y_1+u,y_1+u,y_1,s)\; A_o(y_1+u,y_1) \quad . \tag{6.11d}$$

Keeping in mind that u is the distance from the line x = y and is thus always positive, we now show that

$$A_o(y_1+u,y_1) = A_o(x_1,x_1-u) \quad , \tag{6.12a}$$

$$R_R(y_1+u,y_1+u,y_1,s) = R_R(x_1,x_1,x_1-u,s) \quad , \tag{6.12b}$$

$$R_R(y_1,y_1+u,y_1,s) = R_R(x_1-u,x_1,x_1-u,s) \quad , \tag{6.12c}$$

$$R_L(y_1+u,y_1+u,y_1,s) = R_L(x_1,x_1,x_1-u,s) \quad , \tag{6.12d}$$

$$R_L(y_1,y_1+u,y_1,s) = R_L(x_1-u,x_1,x_1-u,s) \quad . \tag{6.12e}$$

Looking at the figure we see that the points (x_1,x_1-u) and (y_1+u,y_1) both lie on the same line parallel to the line x = y. So by Corollary 6.3 equation (6.12a) is true. The remaining four equations follow from the fact that the points lie on the same line and Corollary 6.2.

Now rewriting equation (6.11) using the relations (6.12) we obtain

$$\frac{\partial}{\partial u} R_R(y_1,y_1+u,y_1,s) = -a(s)\, R_R(y_1,y_1+u,y_1,s)$$

$$+ c\, R_R(y_1+u,y_1+u,y_1,s)\; A_o(y_1+u,y_1) \quad , \tag{6.13a}$$

$$\frac{\partial}{\partial u} R_R(y_1+u,y_1+u,y_1,s) = c\, R_R(y_1,y_1+u,y_1,s)\; A_o(y_1+u,y_1) \quad , \tag{6.13b}$$

$$\frac{\partial}{\partial u} R_L(y_1+u,y_1+u,y_1,s) = -a(s) \quad R_L(y_1+u,y_1+u,y_1,s)$$

$$+ c\ R_L(y_1,y_1+u,y_1,s)\ A_o(y_1+u,y_1) \quad , \qquad (6.14a)$$

$$\frac{\partial}{\partial u} R_L(y_1,y_1+u,y_1,s) = c\ R_L(y_1+u,y_1+u,y_1,s)\ A_o(y_1+u,y_1) \quad . \qquad (6.14b)$$

The initial conditions are still

$$R_R(\xi,\xi,\xi,s) = 1 \quad , \qquad (2.30)$$

$$R_L(\xi,\xi,\xi,s) = 1 \quad . \qquad (2.31)$$

Now the equation set (6.13) with s fixed has a unique solution with the initial condition (2.30). Similarly the set (6.14) has a unique solution with the same initial value [see equation (2.31)]. But the two equation sets are identical.[It must be remembered that the first argument only serves as an identifier which distinguishes between, for example, $R_R(x,x,y,s)$ and $R_R(y,x,y,s)$.] Since the equation sets also have identical starting values, we conclude that, for the same s, the solution sets of these equations must also be identical. This implies that

$$R_R(y_1,x_1,y_1,s) = R_L(x_1,x_1,y_1,s) \quad ,$$

$$R_R(x_1,x_1,y_1,s) = R_L(y_1,x_1,y_1,s) \quad .$$

Since this is true for an arbitrary point, we have proved the theorem.∎

Corollary 6.5. For $\gamma(z) = c$, a constant,

$$A_i(x,y) = B_i(x,y) \quad , \qquad (6.15a)$$

$$C_i(x,y) = D_i(x,y) \quad . \qquad (6.15b)$$

Proof: Using the definition of $A_i(x,y)$ and the theorem, we see that

$$A_i(x,y) = \int_0^\infty k(s') \; a^i(s') \; R_R(y,x,y,s') \; ds'$$

$$= \int_0^\infty k(s') \; a^i(s') \; R_L(x,x,y,s') \; ds'$$

which is equal to $B_i(x,y)$. Equation (6.15b) is proved similarly.∎

We now define new variables, T and S, so chosen that the specialized imbedding moment equations and the specialized imbedding equations are dependent on the single variable T, independent of S. We choose to define T by

$$T = x-y \quad . \tag{6.16}$$

The coordinate S, orthogonal to T, is obtained from the condition

$$\nabla T \cdot \nabla S = 0 \tag{6.17}$$

or

$$\frac{\partial T}{\partial x}\frac{\partial S}{\partial x} + \frac{\partial T}{\partial y}\frac{\partial S}{\partial y} = 0 \quad . \tag{6.18}$$

Using equation (6.16), we have

$$\frac{\partial S}{\partial x} - \frac{\partial S}{\partial y} = 0 \quad . \tag{6.19}$$

which is satisfied by the function (although by no means uniquely)

$$S = e^{x+y} \quad . \tag{6.20}$$

Lemma 6.6 For $\gamma(z) = c$, a constant, the specialized imbedding moment equations (3.6) and the specialized imbedding equations (2.34) are functions of the single variable $T = x - y$.

Proof: Since

$$\frac{\partial}{\partial x} = \frac{\partial T}{\partial x}\frac{\partial}{\partial T} + \frac{\partial S}{\partial x}\frac{\partial}{\partial S} \text{ and } \frac{\partial}{\partial y} = \frac{\partial T}{\partial y}\frac{\partial}{\partial T} + \frac{\partial S}{\partial y}\frac{\partial}{\partial S} \quad ,$$

we can write the specialized imbedding moment equations in the form

$$\frac{\partial T}{\partial x}\frac{\partial A_i}{\partial T} + \frac{\partial S}{\partial x}\frac{\partial A_i}{\partial S} = -A_{i+1} + cC_iA_o \quad ,$$

$$\frac{\partial T}{\partial y}\frac{\partial C_i}{\partial T} + \frac{\partial S}{\partial y}\frac{\partial C_i}{\partial S} = -cA_iA_o \quad ,$$

$$\frac{\partial T}{\partial x}\frac{\partial D_i}{\partial T} + \frac{\partial S}{\partial x}\frac{\partial D_i}{\partial S} = cB_iA_o \quad ,$$

$$\frac{\partial T}{\partial y}\frac{\partial B_i}{\partial T} + \frac{\partial S}{\partial y}\frac{\partial B_i}{\partial S} = B_{i+1} - cD_iA_o \quad , \qquad i = 0,1,2, \ldots .$$

Here we have used Theorem 4.2 to set $B_o(x,y) = A_o(x,y)$. From equations (6.16) and (6.20) we have

$$\frac{\partial T}{\partial x} = 1 \quad ; \quad \frac{\partial T}{\partial y} = -1 \quad ; \quad \frac{\partial S}{\partial x} = S \quad ; \quad \frac{\partial S}{\partial y} = S \quad .$$

Substituting these values into the set of equations above gives

$$\frac{\partial A_i}{\partial T} + S\frac{\partial A_i}{\partial S} = -A_{i+1} + cC_iA_o \quad , \tag{6.21a}$$

$$\frac{-\partial C_i}{\partial T} + S\frac{\partial C_i}{\partial S} = -cA_iA_o \quad , \tag{6.21b}$$

$$\frac{\partial D_i}{\partial T} + S\frac{\partial D_i}{\partial S} = cB_iA_o \quad , \tag{6.21c}$$

$$\frac{-\partial B_i}{\partial T} + S\frac{\partial B_i}{\partial S} = B_{i+1} - cD_iA_o \quad , \qquad i = 0,1,2, \ldots . \tag{6.21d}$$

Now if a moment is independent of the coordinate S, its partial derivative with respect to S will be zero. Adding equations (6.21a) and

(6.21d) and using Corollary 6.5, we find that

$$2 S \frac{\partial A_i}{\partial S} = 0 \quad .$$

Since $S = e^{x+y}$, S cannot be zero and hence

$$\frac{\partial A_i}{\partial S} = \frac{\partial B_i}{\partial S} = 0 \quad .$$

Adding equations (6.21b) and (6.21c) and using Corollary 6.5 yields

$$2 S \frac{\partial C_i}{\partial S} = 0 \quad ,$$

from which it follows that

$$\frac{\partial C_i}{\partial S} = \frac{\partial D_i}{\partial S} = 0 \quad .$$

This shows that the specialized imbedding moment equations are independent of S.

The proof of the second assertion of the theorem is made in a similar way using the R function equalities shown in Theorem 6.4. ∎

If we now affect the transformation to the variable T in the specialized imbedding moment equations and in the specialized imbedding equations we arrive at the sets

$$\frac{d}{dT} A_i(T) = - A_{i+1}(T) + cC_i(T) A_0(T) \quad , \tag{6.22a}$$

$$\frac{d}{dT} C_i(T) = c A_i(T) A_0(T) \quad , \tag{6.22b}$$

$$A_i(0) = A_i(\xi,\xi) = \int_0^\infty k(s') a^i(s') ds' \quad , \tag{6.22c}$$

$$C_i(0) = C_i(\xi,\xi) = \int_0^\infty k(s') \, a^i(s') \, ds' \quad , \; i = 0,1,2,\ldots, \qquad (6.22d)$$

$$\frac{\partial}{\partial T} R_{Rx}(T,s) = c \, R_{Ry}(T,s) \, A_0(T) \quad , \qquad (6.23a)$$

$$\frac{\partial}{\partial T} R_{Ry}(T,s) = -a(s) \, R_{Ry}(T,s) + c \, R_{Rx}(T,s) \, A_0(T) \quad , \qquad (6.23b)$$

$$R_{Rx}(0,s) = R_R(\xi,\xi,\xi,s) = 1 \quad , \qquad (6.23c)$$

$$R_{Ry}(0,s) = R_R(\xi,\xi,\xi,s) = 1 \quad . \qquad (6.23d)$$

Here we have made the reidentifications

$$R_{Rx}(T,s) = R_R(x,x,y,s) \quad , \qquad (6.23)$$

$$R_{Ry}(T,s) = R_R(y,x,y,s) \quad . \qquad (6.25)$$

In the process of transforming these equations we have reduced them in number by half because of the equalities shown in Theorem 6.4 and its Corollary 6.5. Equations (6.22) and (6.23) are now in the form of an initial value problem. This is the form used to compute the examples in the section on numerics. We have not yet shown the existence and uniqueness of solutions to these sets. We now carry out a further transformation which allows the application of the results in [3].

We first write equations (6.22) and (6.23) in matrix form:

$$\frac{d}{dT}\begin{pmatrix} A_i \\ C_i \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}\begin{pmatrix} A_i \\ C_i \end{pmatrix} A_0 - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_{i+1} \\ C_{i+1} \end{pmatrix} \quad , \qquad (6.26)$$

$$\frac{\partial}{\partial T}\begin{pmatrix} R_{Ry} \\ R_{Rx} \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}\begin{pmatrix} R_{Ry} \\ R_{Rx} \end{pmatrix} A_0 - \begin{pmatrix} a(s) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} R_{Ry} \\ R_{Rx} \end{pmatrix} \quad . \qquad (6.27)$$

If we define the function f(T) by

$$f(T) = c \int_0^T A_0(t)\, dt \qquad (6.28)$$

and assume for the moment that it is known, then the two vectors

$$\begin{pmatrix} \cosh f \\ \sinh f \end{pmatrix} \quad , \quad \begin{pmatrix} \sinh f \\ \cosh f \end{pmatrix}$$

each satisfy the homogeneous equations associated with equations (6.26) and (6.27). For example, since

$$\frac{d}{dT} f(T) = c\, A_0(T) \quad , \qquad (6.29)$$

we have

$$\frac{d}{dT} \begin{pmatrix} \cosh f \\ \sinh f \end{pmatrix} = \begin{pmatrix} \sinh f \\ \cosh f \end{pmatrix} c\, A_0 \quad ,$$

$$= \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \begin{pmatrix} \cosh f \\ \sinh f \end{pmatrix} A_0 \quad .$$

The matrix

$$D\big(f(T)\big) = \begin{pmatrix} \cosh f & \sinh f \\ \sinh f & \cosh f \end{pmatrix} \qquad (6.30)$$

is a fundamental solution matrix for the homogeneous parts of equations (6.26) and (6.27). Using this matrix we define an infinite sequence of vectors $\begin{pmatrix} U_i \\ V_i \end{pmatrix}$ through the relations

$$\begin{pmatrix} A_i \\ C_i \end{pmatrix} = D(f) \begin{pmatrix} U_i \\ V_i \end{pmatrix} \quad , \qquad i = 0,1,2... \qquad (6.31)$$

and define the vector $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ by

$$\begin{pmatrix} R_{Ry} \\ R_{Rx} \end{pmatrix} = D(f) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad . \tag{6.32}$$

Applying a variation of parameters technique to the systems (6.26) and (6.27), we can show that the vectors $\begin{pmatrix} U_i \\ V_i \end{pmatrix}$ satisfy the equations

$$\frac{d}{dT} \begin{pmatrix} U_i(T) \\ V_i(T) \end{pmatrix} = B\Big(f(T)\Big) \begin{pmatrix} U_{i+1}(T) \\ V_{i+1}(T) \end{pmatrix} \quad , \tag{6.33a}$$

$$\frac{df(T)}{dT} = U_0 \cosh f(T) + V_0 \sinh f(T) \quad , \tag{6.33b}$$

$$f(0) = 0 \quad , \tag{6.33c}$$

$$U_i(0) = V_i(0) = \int_0^\infty k(s') \, a^i(s') \, ds' \, , \qquad i = 0,1,2,\ldots, \tag{6.33d}$$

and that the vector $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ satisfies

$$\frac{d}{dT} \begin{pmatrix} z_1(T) \\ z_2(T) \end{pmatrix} = a(s) \, B\Big(f(T)\Big) \begin{pmatrix} z_1(T) \\ z_2(T) \end{pmatrix} \quad , \tag{6.34a}$$

$$z_1(0) = z_2(0) = 1 \quad . \tag{6.34b}$$

where the matrix $B\Big(f(T)\Big)$ is defined by

$$B\Big(f(T)\Big) = \begin{pmatrix} -\cosh^2 f(T) & -\sinh f(T) \cosh f(T) \\ \sinh f(T) \cosh f(T) & \sinh^2 f(T) \end{pmatrix} \tag{6.34c}$$

Allen and Kyner [3] have given a local existence and uniqueness proof for systems of the type (6.33). Specifically, they show that systems of this type are equivalent to the functional differential equation

$$\frac{df}{dt} = F(f,t) \; Ⓗ (t,t_0;f) \quad , \quad t \; \varepsilon [t_0,t_1] \quad , \quad f(t_0) = f_0 \quad ,$$

with

$$Ⓗ (t,t_0;f) = \sum_{k=0}^{\infty} \square^{(k)} (t,t_0;f) \; \alpha_k \quad ,$$

$$\square^{(0)} = E, \text{ the identity matrix} \quad ,$$

$$\square^{(k+1)} (t,t_0;f) = \int_{t_0}^{t} B\Big(f(t')\Big) \; \square^{(k)} (t',t_0;f) \; dt' \quad .$$

The α_k are finite dimensional vectors related to the initial conditions of the equivalent problem of the type (6.33). They prove for appropriate norm and appropriate α_k

Theorem 6.7. Suppose $||B\Big(f(T)\Big)|| \le K$ and let $[t_0,t_1]$ be short enough so that

$$\sum_{j=0}^{\infty} \Big(K(t_1-t_0)\Big)^j \; \frac{||\alpha_j||}{j!} < \infty \quad .$$

Then there exists a unique solution to the equation

$$\frac{df}{dt} = F(f,t) \; Ⓗ \; (t,t_0;f) \quad , \quad f(t_0) = f_0 \quad , \quad t \; \varepsilon [t_0,t_1] \quad .$$

Proof: See [3].

For our problem

$$\alpha_j = \begin{pmatrix} U_j(0) \\ V_j(0) \end{pmatrix} \quad ,$$

$$F(f,T) = [\cosh f(T), \quad \sinh f(T)] \quad .$$

The conditions of Theorem 6.7 are fulfilled because the norm of the matrix (6.34c) is easy to bound if T is restricted and the condition on the sum is a consequence of Corollary 5.2. In order to show the latter we note that we may take for the norm on α_j

$$||\alpha_j|| = \text{Max} \left\{ |U_j(0)| \ , \ |V_j(0)| \right\} = \left| \int_0^\infty k(s') \ a^j(s') \ ds' \right| \quad .$$

The radius of convergence, R, of the series in the statement of the theorem is then determined from

$$\frac{1}{R} = \limsup_{j \to \infty} \left(\frac{K^j \displaystyle\int_0^\infty k(s') \ a^j(s') \ ds'}{j!} \right)^{1/j}$$

$$= K \limsup_{j \to \infty} \left(\frac{\displaystyle\int_0^\infty k(s') \ a^j(s') \ ds'}{j!} \right)^{1/j}$$

so that

$$\frac{1}{R} \leq \frac{K}{X - Y}$$

by Corollary 5.2. Since X - Y is non-zero we are assured that the series of the theorem has a non-vanishing radius of convergence, and hence, the specialized imbedding moment equations have a unique solution.

It is remarked in [3] that the moment method probably cannot be applied in the absence of a growth constraint on the initial conditions α_j. This appears to be the case since, if restriction R3 is omitted, a counter example can be given which shows Theorem 5.1 invalid.

Given the unique solution assured by the theorem, the function f(T) is known and the existence and uniqueness of a solution to equation (6.26) follows from the theory of ordinary differential equations. It is further shown in [3] that the solutions of the truncated system will converge to the solution of (6.26). This completes the justification of the moment algorithm as it applies to the specialized imbedding moment equations. Given the solution to these equations, the existence and uniqueness of a solution to the specialized imbedding equations then follows via equation (6.32).

It remains to prove the existence and uniqueness of a solution to the imbedding moment equations and also to show that the solution of the truncated problem converges to the solution of the infinite set.

The infinite set of imbedding moment equations

$$G_i' = -G_{i+1} + c\, C_i\, G_0 \qquad i = 0,1,2,\ldots$$

can be written in the form

$$\Gamma' = -\sigma\Gamma + f(t) \tag{6.35}$$

where Γ and f(t) are the infinite vectors

$$\Gamma = \begin{pmatrix} G_0 \\ G_1 \\ G_2 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \tag{6.36}$$

and

$$f(t) = \begin{pmatrix} C_0(t) \\ C_1(t) \\ C_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} cG_0(t) \quad , \tag{6.37}$$

and σ is the infinite matrix

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} \quad . \tag{6.38}$$

Let η be a fundamental matrix for the system (6.35); then η can be written in the form [5]

$$\eta = e^{-\sigma t} \tag{6.39}$$

Using this, we define a new variable Θ by the transformation

$$\Gamma = e^{-\sigma t}\Theta \quad . \tag{6.40}$$

Then

$$\begin{aligned} \Gamma' &= -\sigma e^{-\sigma t}\Theta + e^{-\sigma t}\Theta' \\ &= -\sigma\Gamma + e^{-\sigma t}\Theta' \quad . \end{aligned}$$

Using equation (6.35) we find

$$e^{-\sigma t}\Theta' = f(t)$$

or

$$\Theta' = e^{\sigma t}f(t) \quad , \tag{6.41}$$

where $e^{\sigma t}$ is the inverse of $e^{-\sigma t}$ and is given by

$$e^{\sigma t} = I + \sigma t + \frac{\sigma^2 t^2}{2!} + \frac{\sigma^3 t^3}{3!} + \cdots \quad . \tag{6.42}$$

The solution to equation (6.35) is then given by

$$\Gamma = e^{-\sigma t}\Gamma_0 + e^{-\sigma t}\int_0^t e^{\sigma w}f(w)\ dw \quad . \tag{6.43}$$

We now isolate the first component of equation (6.43). First we find the product of $e^{-\sigma t}e^{\sigma w}$:

$$e^{-\sigma t} e^{\sigma w} = \begin{pmatrix} 1 & -t & \frac{t^2}{2!} & -\frac{t^3}{3!} & \cdot & \cdot \\ 0 & 1 & -t & \frac{t^2}{2!} & \cdot & \cdot \\ 0 & 0 & 1 & -t & \cdot & \cdot \\ 0 & 0 & 0 & 1 & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots\vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 & w & \frac{w^2}{2!} & \frac{w^3}{3!} & \cdot & \cdot \\ 0 & 1 & w & \frac{w^2}{2!} & \cdot & \cdot \\ 0 & 0 & 1 & w & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & w-t & \frac{(w-t)^2}{2!} & \frac{(w-t)^3}{3!} & \cdots \\ 0 & 1 & w-t & \frac{(w-t)^2}{2!} & \cdots \\ 0 & 0 & 1 & w-t & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots \end{pmatrix}$$

$$= e^{\sigma(w-t)}$$

The integrand of equation (6.43) is thus

$$e^{\sigma(w-t)} f(w) = \begin{pmatrix} 1 & w-t & \frac{(w-t)^2}{2!} & \cdots \\ 0 & 1 & w-t & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \end{pmatrix} c\, G_0$$

$$= \begin{pmatrix} c\, G_0 \sum_{j=0}^{\infty} \frac{C_j (w-t)^j}{j!} \\ \vdots \\ c\, G_0 \sum_{j=0}^{\infty} \frac{C_{j+n-1} (w-t)^j}{j!} \\ \vdots \end{pmatrix} .$$

From (6.43) and this result,

$$G_0 (t) = \sum_{j=0}^{\infty} (-1)^j \frac{C_j(0)}{j!} t^j + c \int_0^t \left[\sum_{j=0}^{\infty} \frac{C_j(w)(w - t)^j}{j!} \right] G_0(w)\, dw . \quad (6.44)$$

This is a Volterra integral equation of the second kind in G_0 with kernel

$$\tilde{K}(t,w) = \sum_{j=0}^{\infty} \frac{C_j(w)(w - t)^j}{j!} . \quad (6.45)$$

It is a consequence of Corollary 5.3 that the series for $\tilde{K}(t,w)$ has radius of convergence at least X - Y. Since we are concerned only with values of (w - t) which are less than this value, we can assert that $\tilde{K}(t,w)$ is bounded. From the theory of Volterra integral equations [10] we can then conclude that (6.44) has a unique solution. Given this unique G_0, the remaining G_i are also unique for they are determined iteratively by the relations

$$
\begin{aligned}
G_1 &= -G_0' + c\, C_0\, G_0 \\
&\;\;\vdots \\
G_{i+1} &= -G_i' + c\, C_i\, G_0 \quad . \\
&\;\;\vdots
\end{aligned}
$$

We now show that when we truncate the systems (6.35), the solutions to the truncated system approach the solution of (6.35). We treat only the convergence of G_0 since this is the only moment we require. At the end of the proof we indicate how the proof can be extended to all the G_i. By carrying out the equivalent procedure on the truncated system, we find that

$$\Theta_n' = e^{\sigma_n t} f_n(t) \tag{6.46}$$

where the n indicates that the system (6.35) was truncated by setting G_{n+i} equal to zero for i equal to 1,2,... etc. The solution to the truncated system is thus given by

$$\Gamma_n = e^{-\sigma_n t}\Gamma_{0,n} + e^{-\sigma_n t}\int_0^t e^{\sigma_n w} f_n(w)\, dw \tag{6.47}$$

where, as before,

$$e^{-\sigma_n t} e^{\sigma_n w} = e^{\sigma_n (w-t)}$$

and

$$e^{-\sigma_n (w-t)} f_n(t) = \begin{pmatrix} c\, G_{n,0} \sum_{j=0}^{n} \frac{C_j (w-t)^j}{j!} \\ c\, G_{n,0} \sum_{j=0}^{n-1} \frac{C_{j+n-1} (w-t)^j}{j!} \\ \vdots \\ c\, G_{n,0}\, C_{n-1} \end{pmatrix} .$$

From this we find

$$G_{n,0} = \sum_{j=0}^{n} \frac{(-1)^j C_j(0)\, t^j}{j!} + c \int_0^t \left(\sum_{j=0}^{n} \frac{C_j(w)(w-t)^j}{j!} \right) G_{n,0}(w)\, dw \tag{6.48}$$

so that

$$\begin{aligned}(G_0 - G_{n,0}) &= \sum_{n+1}^{\infty} \frac{(-1)^j C_j(0) t^j}{j!} \\ &+ c\int_0^t \sum_{j=0}^{n} \frac{C_j(w)(w-t)^j}{j!} \left[G_0(w) - G_{n,0}(w)\right] dw \\ &+ c\int_0^t \sum_{j=n+1}^{\infty} \frac{C_j(w)(w-t)^j}{j!} G_0(w)\, dw \quad .\end{aligned} \tag{6.49}$$

In this equation the infinite series which appear are the tails of uniformly convergent series by Corollaries 5.2 and 5.3. Also $G_0(t)$ is bounded in [0,t]. Therefore, letting n approach infinity we conclude that the difference $G_0 - G_{n,0}$ approaches the solution of

$$y(t) = \int_0^t \sum_{j=0}^{\infty} \frac{C_j(w)(w-t)^j}{j!} \; y(w) \; dw \quad . \tag{6.50}$$

This is also a Volterra integral equation of the second kind with $f(t) = 0$; hence, it has the unique solution

$$y(t) = c\int_0^t \tilde{K}(t,w) \; f(w) \; dw = 0 \quad ,$$

showing that

$$G_{n,0} \to G_0 \text{ as } n \to \infty \quad .$$

We have proved the following theorem.

<u>Theorem 6.8</u>. Under the same hypotheses for which Theorem 5.1 holds the imbedding-moment equations, (3.11), have a unique solution and the zeroth order moment, $G_{n,0}$, from the solution of the truncated imbedding-moment equations converges to the zeroth order moment, G_0, of the solution of the imbedding-moment equations.

For the remaining moments we can arrive at equations similar to (6.49). The argument used above can be made for each of these equations provided we can make the infinite series appearing in the first and last terms arbitrarily small. These series involve C_{j+k} instead of C_j and can be majorized by the derivatives of the series

$$\sum_{j=0}^{\infty} \left(\underset{[0,t]}{\text{Max}} \; |C_j| \right) \frac{t^j}{j!} \tag{6.51}$$

That the series (6.51) converges for $t < X - Y$ follows from the proof of Theorem 5.1 and the definition of the moments, C_j. Since it is absolutely and uniformly convergent inside its circle of convergence the series (6.51) may be differentiated term by term and the resulting series will also converge for $t < X - Y$.

This completes the justification of the method of moments.

VII CASE WHERE $\gamma(z)$ IS A STEP FUNCTION

We approach the case where $\gamma(z)$ is a step function by first considering the problem when $\gamma(z)$ has a single discontinuity. This is then extended to multiple steps. We attack the single step problem by considering it as the juxtaposition of two problems of the type treated in the last section, one for each value of $\gamma(z)$. Results are derived as if $\gamma(z)$ were non-constant wherever possible. For clarity we will discuss the problem as if it were a transport problem in a double slab. However, this in no way affects its generality.

From the pseudo-transport equation correspondence, we know that there exist left and right reflection and transmission operators and that the pseudo flux, $N(z,x,y,s)$, can be written in terms of these operators. We will examine these operators for a slab extending from y to x. We introduce the quantities u and v defined by

$$u(z,x,y,s) = N(z,x,y,s) \quad , \quad s > 0 \quad , \tag{7.1}$$

$$v(z,x,y,s) = N(z,x,y,s) \quad , \quad s < 0 \quad . \tag{7.2}$$

Thus $v(z,x,y,s)$ and $u(z,x,y,s)$ represent the left-moving and right-moving fluxes at the point z in a slab with left edge at y and right edge at x. From equation (2.11) we have

$$u(z,x,y,s) = k(s) \int_y^z \gamma(z') \quad \eta(z',x,y) \; e^{a(s)(z'-z)} \, dz'$$

$$+ \; h(s) \; e^{a(s)(y-z)} \quad , \qquad s > 0 \quad , \tag{7.3}$$

$$v(z,x,y,s) = k(s) \int_z^x \gamma(z') \ \eta(z',x,y) \ e^{a(s)(z-z')} \ dz'$$

$$+ f(s) \ e^{a(s)(z-x)} \quad , \qquad s < 0 \quad . \tag{7.4}$$

In these equations h(s) is the right-moving flux entering the slab at y, while f(s) is the left-moving flux entering at x. $\eta(z,x,y)$ is the solution of the integral equation (1.1) when the limits of integration correspond to y and x. This solution is given in terms of f(s), h(s) and the R functions by equation (2.17),

$$\eta(z,x,y) = \int_{-\infty}^{0} f(s') \ R_R(z,x,y,s') \ ds' + \int_0^{\infty} h(s') \ R_L(z,x,y,s') \ ds' \ . \tag{2.17}$$

When equation (2.17) is substituted into equations (7.3) and (7.4), we find, after changing the order of integration,

$$u(z,x,y,s) = k(s) \int_{-\infty}^{0} f(s') \ ds' \int_y^z \gamma(z') \ e^{a(s)(z'-z)} \ R_R(z',x,y,s') \ dz'$$

$$+ k(s) \int_0^{\infty} h(s') \ ds' \int_y^z \gamma(z') \ e^{a(s)(z'-z)} \ R_L(z',x,y,s') \ dz'$$

$$+ h(s) \ e^{a(s)(y-z)} \quad , \tag{7.5}$$

$$v(z,x,y,s) = k(s) \int_{-\infty}^{0} f(s') \ ds' \int_z^x \gamma(z') \ e^{a(s)(z-z')} \ R_R(z',x,y,s') \ dz'$$

$$+ f(s) \ e^{a(s)(z-x)} \quad ,$$

$$+ k(s) \int_0^{\infty} h(s') \ ds' \int_z^x \gamma(z') \ e^{a(s)(z-z')} \ R_L(z',x,y,s') \ dz' . \tag{7.6}$$

We consider the slab split into two slabs at z. The left-hand slab extends from y to z and the right-hand slab from z to x. We now have four expressions from equations (7.5) and (7.6) for the fluxes at the point z:

$$u_\ell(z,z,y,s) = k(s)\int_{-\infty}^{0} f_1(s') \, ds' \int_y^z \gamma(z') \, e^{a(s)(z-z')} R_R(z',z,y,s') \, dz'$$

$$+ k(s)\int_0^{\infty} h(s') \, ds' \int_y^z \gamma(z') \, e^{a(s)(z'-z)} R_L(z',z,y,s') \, dz'$$

$$+ h(s) \, e^{a(s)(y-z)} \quad , \tag{7.7a}$$

$$u_r(z,x,z,s) = h_2(s) \quad , \tag{7.7b}$$

$$v_\ell(z,z,y,s) = f_1(s) \quad , \tag{7.7c}$$

$$v_r(z,x,z,s) = k(s)\int_{-\infty}^{0} f(s') \, ds' \int_z^x \gamma(z') \, e^{a(s)(z-z')} R_R(z',x,z,s') \, dz'$$

$$+ f(s) \, e^{a(s)(z-x)}$$

$$+ k(s)\int_0^{\infty} h_2(s') \, ds' \int_z^x \gamma(z') \, e^{a(s)(z-z')} R_L(z',x,z,s') \, dz' \, . \tag{7.7d}$$

Here $f_1(s)$ is the flux entering the left-hand slab from the right, and $h_2(s)$ is the flux entering the right-hand slab from the left.

Transport theory allows us to write two further expressions for the fluxes at z in terms of the reflection and transmission operators.

$$u_{\ell}(z,z,y,s) = \mathcal{T}_{\ell}(z,y,s) \circ h(s) + \mathcal{R}_{r}(z,y,s) \circ v_{\ell}(z,z,y,s) \quad , \qquad (7.8a)$$

$$v_{r}(z,x,z,s) = \mathcal{R}_{\ell}(z,x,s) \circ u_{r}(z,x,z,s) + \mathcal{T}_{r}(z,x,s) \circ f(s) \quad . \qquad (7.8b)$$

These equations are the statement of the principles of invariance for a finite slab. The principles are shown by Chandrasekhar in [4]. By comparing these equations with (7.7), we can write expressions for the operators appearing in equations (7.8)

$$\mathcal{T}_{\ell}(z,y,s) \circ = k(s) \int_0^{\infty} \Big(\quad (s') \Big) ds' \int_y^z \gamma(z') \, e^{a(s)(z'-z)} \, R_L(z',z,y,s') \, dz'$$

$$+ \, (\quad) \circ e^{a(s)(y-z)} \quad , \qquad (7.9a)$$

$$\mathcal{R}_{r}(z,y,s) \circ = k(s) \int_{-\infty}^{0} \Big(\quad (s') \Big) ds' \int_y^z \gamma(z') \, e^{a(s)(z'-z)} \, R_R(z',z,y,s') \, dz' \, ,$$

$$(7.9b)$$

$$\mathcal{R}_{\ell}(z,x,s) \circ = k(s) \int_0^{\infty} \Big(\quad (s') \Big) ds' \int_z^x \gamma(z') \, e^{a(s)(z-z')} \, R_L(z',x,z,s') \, dz' \, ,$$

$$(7.9c)$$

$$\mathcal{T}_{r}(z,x,s) \circ = k(s) \int_{-\infty}^{0} \Big(\quad (s') \Big) ds' \int_z^x \gamma(z') \, e^{a(s)(z-z')} \, R_R(z',x,z,s') \, dz' \, ,$$

$$+ \, (\quad) \circ e^{a(s)(z-x)} \qquad (7.9d)$$

The notation $\Big(\quad (s') \Big)$ denotes the change of variable to s'; the action in this case is by integration over s'. The symbol "$(\quad) \circ$" denotes direct multiplication.

At this point it perhaps would be well to indicate how the coupled pair of integral equations (7.8) can be used to solve the problem of a slab when $\gamma(z)$ is a function having a single step. Let z be the point at which the step occurs; then using the results of Section VI, we can compute the R functions appearing in (7.9) since $\gamma(z)$ is constant on each of the two pieces. By the continuity of the flux in the composite slab we must have

$$h_2(s) = u_{\hbar}(z,x,z,s) = u_{\ell}(z,z,y,s) \quad , \qquad (7.10a)$$

$$f_1(s) = v_{\ell}(z,z,y,s) = v_{\hbar}(z,x,z,s) \qquad (7.10b)$$

so equations (7.8) provide a coupled pair of integral equations for $h_2(s)$ and $f_1(s)$. If these equations can be solved, then the solution to the integral equation is given by one of the equations

$$\phi(w) = \int_{-\infty}^{0} f_1(s') \, R_R(w,z,y,s') \, ds' + \int_0^{\infty} h(s') \, R_L(w,z,y,s') \, ds' \qquad (7.11a)$$

or

$$\phi(w) = \int_{-\infty}^{0} f(s') \, R_R(w,x,z,s') \, ds' + \int_0^{\infty} h_2(s') \, R_L(w,x,z,s') \, ds' \qquad (7.11b)$$

depending on which side of the step the point w lies.

The direct solution of (7.8) is formidable. We attempt to simplify these equations by using integral identities. To this end we define

$$\rho(z,y,s_1,s_2) = k(s_1) \int_y^z \gamma(z') \, e^{a(s_1)(z'-z)} \, R_R(z',z,y,s_2) \, dz' \quad , \qquad (7.12)$$

$$t(z,y,s_1,s_2) = k(s_1) \int_y^z \gamma(z') \, e^{a(s_1)(z'-z)} \, R_L(z',z,y,s_2) \, dz' \quad , \quad (7.13)$$

$$r(z,x,s_1,s_2) = k(s_1) \int_z^x \gamma(z') \, e^{a(s_1)(z-z')} \, R_L(z',x,z,s_2) \, dz' \quad , \quad (7.14)$$

$$\tau(z,x,s_1,s_2) = k(s_1) \int_z^x \gamma(z') \, e^{a(s_1)(z-z')} \, R_R(z',x,z,s_2) \, dz' \quad . \quad (7.15)$$

These functions are the kernels of the reflection and transmission operators defined by equations (7.9). Using these definitions we can write the equations for $h_2(s)$ and $f_1(s)$ in the following form:

$$h_2(s) = \int_{-\infty}^0 f_1(s') \, \rho(z,y,s,s') \, ds' + \int_0^\infty h(s') \, t(z,y,s,s') \, ds' + h(s) \, e^{a(s)(y-z)} \quad , \quad (7.16a)$$

$$f_1(s) = \int_{-\infty}^0 f(s') \, \tau(z,x,s,s') \, ds' + \int_0^\infty h_2(s') \, r(z,x,s,s') \, ds' + f(s) \, e^{a(s)(z-x)} \quad . \quad (7.16b)$$

We need to derive integro differential equations for the reflection and transmission kernels of equations (7.12) through (7.15). Before starting these derivations we prove some relations which are needed in the procedure.

We first note that the R functions satisfy the following integral equations

$$R_R(z,x,y,s) = e^{a(s)(z-x)} + \int_y^x K(z,z') \, R_R(z',x,y,s) \, dz' \quad , \qquad (7.17)$$

$$R_L(z,x,y,s) = e^{a(s)(y-z)} + \int_y^x K(z,z') \, R_L(z',x,y,s) \, dz' \quad . \qquad (7.18)$$

Then we can show

<u>Lemma 7.1</u> The R functions and the reflection and transmission kernels are related by

$$R_R(z,z,y,s) = 1 + \int_0^\infty \rho(z,y,s',s) \, ds' \quad , \qquad (7.19)$$

$$R_L(z,z,y,s) = e^{a(s)(y-z)} + \int_0^\infty t(z,y,s',s) \, ds' \quad , \qquad (7.20)$$

$$R_R(z,x,z,s) = e^{a(s)(z-x)} + \int_0^\infty \tau(z,x,s',s) \, ds' \quad , \qquad (7.21)$$

$$R_L(z,x,z,s) = 1 + \int_0^\infty r(z,x,s',s) \, ds' \quad . \qquad (7.22)$$

Proof: From (7.17)

$$R_R(z,z,y,s) = 1 + \int_y^z K(z,z') \; R_R(z',z,y,s)\, dz'$$

$$= 1 + \int_y^z \gamma(z')\, dz' \int_0^\infty k(s') e^{a(s')(z'-z)} \; R_R(z',z,y,s) \; ds'$$

$$= 1 + \int_0^\infty ds' \left[k(s') \int_y^z \gamma(z') \; e^{a(s')(z'-z)} \; R_R(z',z,y,s) \; dz' \right]$$

$$= 1 + \int_0^\infty \rho(z,y,s',s)\, ds'$$

which shows equation (7.19). The remaining three relations are proved analogously. ∎

Lemma 7.2 The reflection kernels satisfy the exchange relations,

$$k(s_2) \; \rho(z,y,s_1,s_2) = k(s_1) \; \rho(z,y,s_2,s_1) \tag{7.23}$$

and

$$k(s_2) \; r(z,x,s_1,s_2) = k(s_1) \; r(z,x,s_2,s_1) \quad . \tag{7.24}$$

<u>Proof</u>: Some manipulation shows that

$$\frac{\rho(z,y,s_1,s_2)}{k(s_1)} = \int_y^z \gamma(z') \, e^{a(s_1)(z'-z)} \, R_R(z',z,y,s_2) \, dz'$$

$$= \int_y^z \gamma(z') \, e^{a(s_1)(z'-z)} \Bigg[e^{a(s_2)(z'-z)}$$

$$+ \int_y^z Q(z',z_2,z,y) \, e^{a(s_2)(z_2-z)} \, dz_2 \Bigg] \, dz'$$

$$= \int_y^z \gamma(z') \, e^{\left[a(s_1) + a(s_2)\right](z'-z)} \, dz'$$

$$+ \int_y^z e^{a(s_1)(z'-z)} \, dz' \int_y^z \gamma(z') \, Q(z',z_2,z,y) \, e^{a(s_2)(z_2-z)} \, dz_2$$

$$= \int_y^z \gamma(z') \, e^{\left[a(s_1)+a(s_2)\right](z'-z)} \, dz'$$

$$+ \int_y^z e^{a(s_1)(z'-z)} \, dz'$$

$$\times \int_y^z \gamma(z_2) \, Q(z_2,z',z,y) \, e^{a(s_2)(z_2-z)} \, dz_2$$

and

$$\frac{\rho(z,y,s_2,s_1)}{k(s_2)} = \int_y^z \gamma(z') \, e^{a(s_2)(z'-z)} \, R_R(z',z,y,s) \, dz'$$

$$= \int_y^z \gamma(z') \, e^{a(s_2)(z'-z)} \left[e^{a(s_1)(z'-z)} \right.$$

$$\left. + \int_y^z Q(z',z_2,z,y) \, e^{a(s_1)(z_2-z)} \, dz_2 \right] dz'$$

$$= \int_y^z \gamma(z') \, e^{\left[a(s_1)+a(s_2)\right](z'-z)} \, dz'$$

$$+ \int_y^z e^{a(s_1)(z_2-z)} \, dz_2$$

$$\times \int_y^z \gamma(z') \, Q(z',z_2,z,y) \, e^{a(s_2)(z'-z)} \, dz' \ .$$

This establishes (7.23). Relation (7.24) is proved similarly. ▮

We now derive equations for $\rho(z,y,s_1,s_2)$ and $t(z,y,s_1,s_2)$. For these two functions we are interested in equations which can be started at $w = y$ and integrated to z. From equation (7.8a) and the definitions of ρ and t, we have

$$u(w,w,y,s) = \int_{-\infty}^{0} f_1(s') \quad \rho(w,y,s,s') \, ds'$$

$$+ \int_{0}^{\infty} h(s') \, t(w,y,s,s') \, ds' + h(s) \, e^{a(s)(y-w)} \quad . \tag{7.25}$$

We set $h(s) = 0$ to obtain

$$u(w,w,y,s_1) = \int_{-\infty}^{0} f_1(s') \rho(w,y,s_1,s') \, ds' \quad . \tag{7.26}$$

Hence,

$$u_1(w,w,y,s_1) + u_2(w,w,y,s_1) = \int_{-\infty}^{0} f_1(s') \frac{\partial}{\partial w} \rho(w,y,s_1,s') \, ds' \quad . \tag{7.27}$$

From the pseudo-transport equation we have

$$u_1(w,w,y,s_1) = -a(s_1) \, u(w,w,y,s_1) + k(s_1) \, \gamma(w) \, \eta(w,w,y) \quad , \tag{7.28}$$

From (2.17) we have, since $h(s) = 0$,

$$\eta(w,w,y) = \int_{-\infty}^{0} f_1(s') \, R_R(w,w,y,s') \, ds' \quad ,$$

and by Lemma 7.1

$$\eta(w,w,y) = \int_{-\infty}^{0} f_1(s') \left[1 + \int_{0}^{\infty} \rho(w,y,s_2,s') \, ds_2 \right] ds' \quad . \tag{7.29}$$

We insert (7.26) and (7.29) into (7.28) to get

$$u_1(w,w,y,s_1) = -a(s_1) \int_{-\infty}^{0} f_1(s') \, \rho(w,y,s_1,s') \, ds'$$

$$+ k(s_1) \, \gamma(w) \int_{-\infty}^{0} f_1(s') \left[1 + \int_0^{\infty} \rho(w,y,s_2,s') \, ds_2 \right] ds' \quad ,$$

and obtain finally

$$u_1(w,w,y,s_1) = \int_{-\infty}^{0} f_1(s') \, ds' \left\{ -a(s_1) \, \rho(w,y,s_1,s') \right. \qquad (7.30)$$

$$\left. + k(s_1) \, \gamma(w) \left[1 + \int_0^{\infty} \rho(w,y,s_2,s') \, ds_2 \right] \right\} \quad . \qquad (7.30)$$

To get an expression for $u_2(w,w,y,s_1)$ we differentiate the pseudo-transport equation with respect to x. It is established in [1] that this operation is permitted and that the order in which the differentiations are done can be interchanged. The result is

$$u_{12}(z,x,y,s) + a(s) \, u_2(z,x,y,s) = k(s) \, \gamma(z) \, \eta_2(z,x,y) \quad ,$$

$$-v_{12}(z,x,y,s) + a(s) \, v_2(z,x,y,s) = k(s) \, \gamma(z) \, \eta_2(z,x,y) \quad ,$$

$$u_2(y,x,y,s) = 0 \quad ,$$

$$v_2(x,x,y,s) = -v_1(x,x,y,s) \quad . \qquad (7.31)$$

Equation (7.31) is a problem exactly analogous to the original transport problem and its solution can be written down immediately from equation (7.5), i.e.

$$u_2(w,w,y,s_1) \doteq k(s_1) \int_{-\infty}^{0} \left[-v_1(w,w,y,s')\right] ds'$$

$$\times \int_{y}^{w} \gamma(z')\; R_R(z',w,y,s')\; e^{a(s_1)(z'-w)}\; dz' \quad ,$$

so by the definition of $\rho(w,y,s_1,s')$

$$u_2(w,w,y,s_1) \doteq -\int_{-\infty}^{0} v_1(w,w,y,s')\; \rho(w,y,s_1 s')\; ds' \quad . \qquad (7.32)$$

The value of $v_1(w,w,y,s\ )$ can be taken from the pseudo-transport equation so

$$u_2(w,w,y,s_1) = \int_{-\infty}^{0} \rho(w,y,s_1,s') \Big[-a(s')\ v(w,w,y,s')$$

$$+ k(s')\ \gamma(w)\ \eta(w,w,y) \Big]\ ds'$$

$$= \int_{-\infty}^{0} \rho(w,y,s_1,s') \Big[-a(s')\ f_1(s')$$

$$+ k(s')\ \gamma(w) \int_{-\infty}^{0} f_1(s_2)\ R_R(w,w,y,s_2) ds_2 \Big]\ ds'$$

$$= \int_{-\infty}^{0} f_1(s') \Big[-a(s')\ \rho(w,y,s_1,s')$$

$$+ \gamma(w)\ R_R(w,w,y,s') \int_{-\infty}^{0} k(s_2)\ \rho(w,y,s_1,s_2)\ ds_2 \Big]\ ds' \quad .$$

Finally using Lemma 7.1

$$u_2(w,w,y,s_1) = \int_{-\infty}^{0} f_1(s') \Bigg\{ -a(s')\ \rho(w,y,s_1,\ s')$$

$$+ \gamma(w) \left[1 + \int_{0}^{\infty} \rho(w,y,s_3,s')\ ds_3 \right]$$

$$\times \int_{-\infty}^{0} k(s_2)\ \rho(w,y,s_1,s_2)\ ds_2 \Bigg\}\ ds' \quad . \qquad (7.33)$$

We put our expressions for the two derivatives into equation (7.27) which results in

$$\int_{-\infty}^{0} f_1(s') \frac{\partial}{\partial w} \rho(w,y,s_1,s')\, ds' = \int_{-\infty}^{0} f_1(s') \Bigg\{ -a(s_1)\, \rho(w,y,s_1,s')$$

$$+ k(s_1)\, \gamma(w) \left[1 + \int_{0}^{\infty} \rho(w,y,s_2,s')\, ds_2 \right] -a(s')\, \rho(w,y,s_1,s')$$

$$+ \gamma(w) \left[1 + \int_{0}^{\infty} \rho(w,y,s_3,s')\, ds_3 \right] \int_{-\infty}^{0} k(s_2)\, \rho(w,y,s_1,s_2)\, ds_2 \Bigg\} ds'$$

The value of $f_1(s)$ is arbitrary as far as the left-hand slab is concerned, hence, we conclude in the usual way that

$$\frac{\partial}{\partial w} \rho(w,y,s_1,s_2) = -\left[a(s_1) + a(s_2)\right] \rho(w,y,s_1,s_2) + \Bigg[k(s_1)\, \gamma(w)$$

$$+ \gamma(w) \int_{-\infty}^{0} k(s_3)\, \rho(w,y,s_1,s_3)\, ds_3 \Bigg] \left[1 + \int_{0}^{\infty} \rho(w,y,s_4,s_2)\, ds_4 \right] ,$$

$$\rho(y,y,s_1,s_2) = 0 \quad . \qquad (7.34)$$

We can reduce this integro differential equation to a differential equation by using Lemmas 7.1 and 7.2. Applying the relation (7.3) yields

$$\frac{\partial}{\partial w}\,\rho(w,y,s_1,s_2) = -\left[a(s_1) + a(s_2)\right]\rho(w,y,s_1,s_2)$$

$$+\; k(s)\,\gamma(w)\left[1 + \int_{-\infty}^{0} \rho(w,y,s_3,s_1)\,ds_3\right]\left[1 + \int_{0}^{\infty} \rho(w,y,s_4,s_2)\right] ds_4 \quad ,$$

and from (7.19)

$$\frac{\partial}{\partial w}\,\rho(w,y,s_1,s_2) = -\left[a(s_1) + a(s_2)\right]\rho(w,y,s_1,s_2)$$

$$+\; k(s_1)\,\gamma(w)\,R_R(w,w,y,s_1)\,R_R(w,w,y,s_2) \quad ,$$

$$\rho(y,y,s_1,s_2) = 0 \quad . \tag{7.35}$$

For y fixed this is an ordinary differential equation with constant coefficients provided the R_R function is known. However, these values are exactly those generated during the integration of the specialized imbedding equations, so at least for $\gamma(z)$ a constant, equation (7.35) can be integrated simultaneously with these equations.

To get a similar equation for $t(z,y,s_1,s_2)$, we again use equation (7.25), but now with $f_1(s)$ equal to 0, so that,

$$u(w,w,y,s_1) = \int_0^\infty h(s')\,t(w,y,s_1,s')\,ds' + h(s_1)\,e^{a(s_1)(y-w)} \tag{7.36}$$

and

$$u_1(w,w,y,s_1) + u_2(w,w,y,s_1) = \int_0^\infty h(s')\,\frac{\partial}{\partial w}\,t(w,y,s_1,s')\,ds'$$

$$-h(s_1)\,a(s_1)\,e^{a(s_1)(y-w)} \quad . \tag{7.37}$$

From the transport equation

$$u_1(w,w,y,s_1) = -a(s_1)\, u(w,w,y,s_1) + k(s_1)\, \gamma(w)\, \eta(w,w,y)$$

$$= -a(s_1)\left[\int_0^\infty h(s')\, t(w,y,s_1,s')\, ds' + h(s_1)\, e^{a(s_1)(y-w)}\right]$$

$$+ k(s_1)\, \gamma(w) \int_0^\infty h(s')\, R_L(w,w,y,s')\, ds'$$

$$= -a(s_1)\, h(s_1)\, e^{a(s_1)(y-w)}$$

$$+ \int_0^\infty h(s') \left[-a(s_1)\, t(w,y,s_1,s') \right.$$

$$\left. + k(s_1)\, \gamma(w)\, R_L(w,w,y,s') \right] ds' \quad ,$$

and from equation (7.20)

$$u_1(w,w,y,s_1) = -a(s_1)\, h(s_1)\, e^{a(s_1)(y-w)} + \int_0^\infty h(s') \left\{ -a(s_1)\, t(w,y,s_1,s\ ) \right.$$

$$\left. + k(s_1)\, \gamma(w) \left[e^{a(s')(y-w)} + \int_0^\infty t(w,y,s_2,s')\, ds_2 \right] \right\} ds' \quad . \tag{7.38}$$

The argument used to derive equation (7.32) still applies, but the value of $v_1(w,w,y,s)$ has changed, hence,

$$u_2(w,w,y,s_1) = -\int_{\infty}^{0} v_1(w,w,y,s') \, \rho(w,y,s_1,s') \, ds'$$

$$= \int_{-\infty}^{0} \rho(w,y,s_1,s') \left[-a(s') \, v(w,w,y,s') + k(s') \, \gamma(w) \, \eta(w,w,y)\right] ds'$$

$$= \int_{-\infty}^{0} \rho(w,y,s_1,s') \, k(s') \, \gamma(w) \left[\int_{0}^{\infty} h(s_2) \, R_L(w,w,y,s_2)\right] ds'$$

since

$$v(w,w,y,s\ ) = f_1(s) = 0$$

This gives finally

$$u_2(w,w,y,s_1) = \int_{0}^{\infty} h(s') \left[e^{a(s')(y-w)} + \int_{0}^{\infty} t(w,y,s_2,s') \quad ds_2\right]$$

$$\times \left[\gamma(w) \int_{-\infty}^{0} k(s_2) \, \rho(w,y,s_1,s_2) \, ds_2\right] ds' \quad . \qquad (7.39)$$

Putting (7.38) and (7.39) into equation (7.37) and cancelling the two non-integral terms we arrive at

$$\int_0^{\infty} h(s') \frac{\partial}{\partial w} t(w,y,s_1,s')\, ds' = \int_0^{\infty} h(s') \Bigg\{ -a(s_1)\, t(w,y,s_1,s')$$

$$+ k(s_1)\, \gamma(w) \left[e^{a(s')(y-w)} + \int_0^{\infty} t(w,y,s_3,s')\, ds_3 \right]$$

$$+ \gamma(w) \left[e^{a(s')(y-w)} + \int_0^{\infty} t(w,y,s_4,s')\, ds_4 \right]$$

$$\times \int_{-\infty}^{0} k(s_5)\, \rho(w,y,s_1,s_5)\, ds_5 \Bigg\}\, ds'$$

And by the usual argument, we have

$$\frac{\partial}{\partial w} t(w,y,s_1,s_2) = -a(s_1)\, t(w,y,s_1,s_2) + \Bigg[e^{a(s_2)(y-w)}$$

$$+ \int_0^{\infty} t(w,y,s_3,s_2)\, ds_3 \Bigg] \left[k(s_1)\, \gamma(w) + \gamma(w) \int_{-\infty}^{0} k(s_5)\, \rho(w,y,s_1,s_5)\, ds_5 \right] . \tag{7.40}$$

Again we can reduce this integro differential equation by using Lemmas 7.1 and 7.2 we obtain

$$\frac{\partial}{\partial w} t(w,y,s_1,s_2) = -a(s_1)\, t(w,y,s_1,s_2)$$

$$+ R_L(w,w,y,s_2)\, k(s_1)\, \gamma(w) \left[1 + \int_{-\infty}^{0} \rho(w,y,s_5,s_1)\, ds_5 \right] .$$

So

$$\frac{\partial}{\partial w} t(w,y,s_1,s_2) = -a(s_1)\ t(w,y,s_1,s_2)$$

$$+ k(s)\ \gamma(w)\ R_L(w,w,y,s_2)\ R_R(w,w,y,s_1) \quad ,$$

$$t(y,y,s_1,s_2) = 0 \quad . \tag{7.41}$$

Like the equation for $\rho(w,y,s_1,s_2)$, these equations can be integrated simultaneously with the specialized imbedding equations.

We now need to obtain similar equations for $r(z,x,s_1,s_2)$ and $\tau(z,x,s_1,s_2)$. Integration of these equations should start at $w = z$ and continue to $w = x$. When the necessary equations are derived we discover that they are integro differential equations as are equations (7.34) and (7.40), but they cannot be reduced by relations similar to those in Lemma 7.2. The interested reader may find the derivations in Appendix B.

It is still possible to solve the problem if we approach it in a different way. We derive equations for r and τ for integration from $w = x$ to $w = z$ and use the symmetry properties of the R functions for constant $\gamma(z)$ to carry out the integrations. The derivations parallel those used to obtain equations (7.34) and (7.40), and the results are

$$\frac{\partial}{\partial w} r(w,x,s_1,s_2) = \left[a(s_1) + a(s_2)\right] r(w,x,s_1,s_2) - \left[1 + \int_0^\infty r(w,x,s_3,s_2)\, ds_3\right]$$

$$\left[k(s_1)\, \gamma(w) + \gamma(w) \int_0^\infty k(s_1)\, r(w,x,s_1,s_4)\, ds_4\right] ,$$

$$r(x,x,s_1,s_2) = 0 \tag{7.42}$$

and

$$\frac{\partial}{\partial w} \tau(w,x,s_1,s_2) = a(s_1)\, \tau(w,x,s_1,s_2)$$

$$- \left[e^{a(s_2)(w-x)} + \int_0^\infty \tau(w,x,s_3,s_2)\, ds_3\right] k(s_1)\, \gamma(w)$$

$$\mathbf{X} \left[1 + \int_0^\infty r(w,x,s',s_1)\, ds'\right] ,$$

$$\tau(x,x,s_1,s_2) = 0 \quad . \tag{7.43}$$

We note that these two equations are identical to (7.34) and (7.40) except for a sign. Using the results of Lemma 7.1 we find that

$$\frac{\partial}{\partial w} r(w,x,s_1,s_2) = \left[a(s_1) + a(s_2)\right] r(w,x,s_1,s_2)$$

$$-k(s_1)\, \gamma(w)\, R_L(w,x,w,s_2)\, R_L(w,x,w,s_1) \tag{7.44}$$

and

$$\frac{\partial}{\partial w} \tau(w,x,s_1,s_2) = a(s_1)\, \tau(w,x,s_1,s_2)$$

$$-k(s_1)\, \gamma(w)\, R_R(w,x,w,s_2)\, R_L(w,x,w,s_1) \quad . \tag{7.45}$$

If we can find a way to get the R function values needed, we can integrate (7.44) and (7.45). We now investigate this problem.

Consider the two slabs shown in the figure below

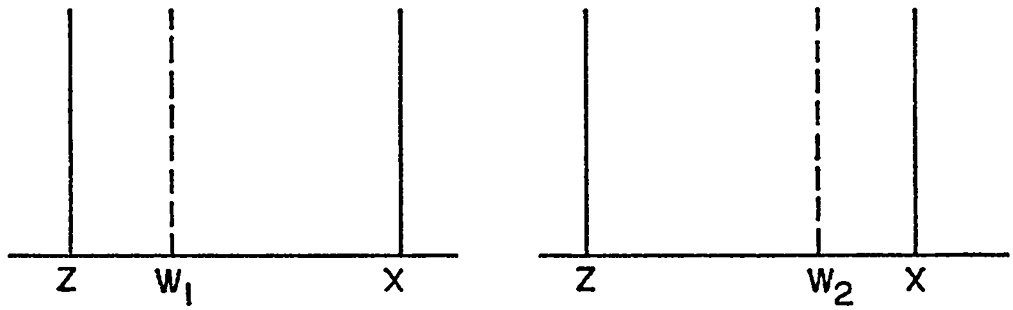

Fig. 3 Integration on w_1 from z to x and on w_2 from x to z.

We assume that

$$w_1 - z = x - w_2 \quad .$$

The slab on the left represents the integration from z to x. This is the direction used in Section VI to compute the R functions. The slab on the right represents the integration from x backwards to z. What we want is to find expressions for $R_L(w_2,x,w_2,s)$ and $R_R(w_2,x,w_2,s)$ in the right-hand slab in terms of the functions $R_L(w_1,w_1,z,s)$ and $R_R(w_1,w_1,z,s)$ which we compute for the left-hand slab. For the case $\gamma(x)$ = constant, the R functions at the edges do not depend on the locations of z, w_1, w_2 and x, but only in the widths of the slabs. Therefore, if

$$w_1 - z = x - w_2 \quad ,$$

then the values at the left edges are the same. This means

$$R_R(z,w_1,z,s) = R_R(w_2,x,w_2,s)$$

and

$$R_L(z,w_1,z,s) = R_L(w_2,x,w_2,s) \quad .$$

Thus, it is possible to integrate the equations for $r(z,x,s_1,s_2)$ and $\tau(z,x,s_1,s_2)$ backwards from x to z while simultaneously integrating the special imbedding equations forward from z to x. When these identifications are made in equations (7.44) and (7.45) and, further the relation from Theorem 6.4,

$$R_R(x,x,y,s) = R_L(y,x,y,s) \quad ,$$

is used we obtain equations identical to the equations for $\rho(z,y,s_1,s_2)$ and $t(z,y,s_1,s_2)$, except for a sign. Since the integration step is negative, we finally find that we really have identical equations; hence, the same algorithm which computes the kernels ρ and t may be used to compute the kernels r and τ.

The extension to the case where $\gamma(z)$ has n steps is now simple. We first solve for the R functions and reflection and transmission kernels for each section of the slab having a constant γ value. We denote the solutions pertaining to the jth value of ρ by the superscript (j). Thus, $R_R^{(j)}$, $\rho^{(j)}$, $t^{(j)}$, etc. refer to the values of R_R, ρ, t in the section of the slab in which γ assumes its jth value. Once the R functions and kernels are computed we then solve for the values of the fluxes at the points of discontinuity of γ. For these fluxes we have the following integral equation set analogous to equations (7.16):

$$f_1(s) = \int_{-\infty}^{0} f_2(s') \, t^{(2)}(z_2,z_1,s,s') \, ds' + \int_{0}^{\infty} h_2(s') \, \rho^{(2)}(z_2,z_1,s,s\ ) \, ds'$$

$$+ f_2(s) \, e^{a(s)(z_1-z_2)} \quad ,$$

$$h_2(s) = \int_{-\infty}^{0} f_1(s') \, \rho^{(1)}(z_1,z_0,s,s') \, ds' + \int_{0}^{\infty} h_1(s') \, t^{(1)}(z_1,z_0,s,s') \, ds'$$

$$+ h_1(s) \, e^{a(s)(z_0-z_1)} \quad ,$$

.

.

.

.

$$f_{n-1}(s) = \int_{-\infty}^{0} f_n(s') \, t^{(n)}(z_n,z_{n-1},s,s') \, ds'$$

$$+ \int_{0}^{\infty} h_n(s') \, \rho^{(n)}(z_n,z_{n-1},s,s') \, ds' + f_n(s) \, e^{a(s)(z_{n-1}-z_n)} \quad ,$$

$$h_n(s) = \int_{-\infty}^{0} f_{n-1}(s') \, \rho^{(n-1)} \, (z_{n-1},z_{n-2},s,s') \, ds'$$

$$+ \int_{0}^{\infty} h_{n-1}(s') \, t^{(n-1)} \, (z_{n-1},z_{n-2},s,s') \, ds'$$

$$+ h_{n-1}(s) \, e^{a(s)(z_{n-s}-z_{n-1})} \quad . \qquad (7.46)$$

In equation (7.46) we have set the known input fluxes to the composite slab equal to $h_1(s)$ and $f_n(s)$ and also

$$r(z,x,s,s') = \rho(x,z,s,s') \quad ,$$

$$\tau(z,x,s,s') = t(x,z,s,s') \quad .$$

After the system (7.46) has been solved, the integral equation solutions are obtained by carrying out the integration:

$$\phi(w) = \int_{-\infty}^{0} f_j(s') \, R_R^{(j)}(w, z_j, z_{j-1}, s') \, ds'$$

$$+ \int_{0}^{\infty} h_j(s') \, R_L^{(j)}(w, z_j, z_{j-1}, s') \, ds' \quad , \qquad (7.47)$$

where it is assumed that

$$z_{j-1} \leq w \leq z_j \quad .$$

In the next section we give some numerical examples of solutions for both constant and step function γ .

VIII NUMERICAL RESULTS

We now turn to the numerical solution of the equations we have derived. We will first give examples of how the computation is carried out for the case

$$\gamma(z) = \text{constant}$$

and then extend to the case where $\gamma(z)$ is a step function.

The computation of the solution of equation (1.1) at the point z by the imbedding moment method takes place in two steps. The first step is to find the R functions at z while the second step is to carry out the integrations of equation (2.17). This latter step is a straight forward application of standard integration techniques. We describe an implementation of the first step in which multiple interior points are calculated for a fixed s. Needless to say all s points could be obtained either by calling the described routine several times or by writing the program to also calculate all s values as well as all z values.

The required values of $R_R(z_1,x,y,s)$ and $R_L(z_1,x,y,s)$ are obtained as follows. The infinite sets of moment equations (6.22a) and (6.22b) are truncated at some predetermined number N and a value of s is chosen. Then starting at y the truncated sets of specialized imbedding moment equations (6.22a) and (6.22b) are integrated to the first interior point, z_1, at which the solution to (1.1) is desired. At this point we also begin to integrate a truncated set of imbedding moment equations (3.11) along with the imbedding equations (3.12a) and (3.12c). Here we should note that it is unnecessary to compute $R_L(z,z,y,s)$ since this value can be obtained from the values computed for $R_R(y,z,y,s)$ by using

Theorem 6.4. The initial conditions required for the new equations are taken from the present values of $C_0(z_1,y)$, $R_R(z_1,z_1,y,s)$ and $R_L(z_1,z_1,y,s)$ which are available from the solutions of the original equation sets. At each succeeding z point, z_i, we pick up a new truncated set of G moment equations and a new pair of imbedding equations obtaining the initial conditions from the current values of $C_0(z_i,y)$, $R_R(z_i,z_i,y,s)$ and $R_L(z_i,z_i,y,s)$. The integration is complete when the desired x value is reached. A flow chart of the calculation is given below

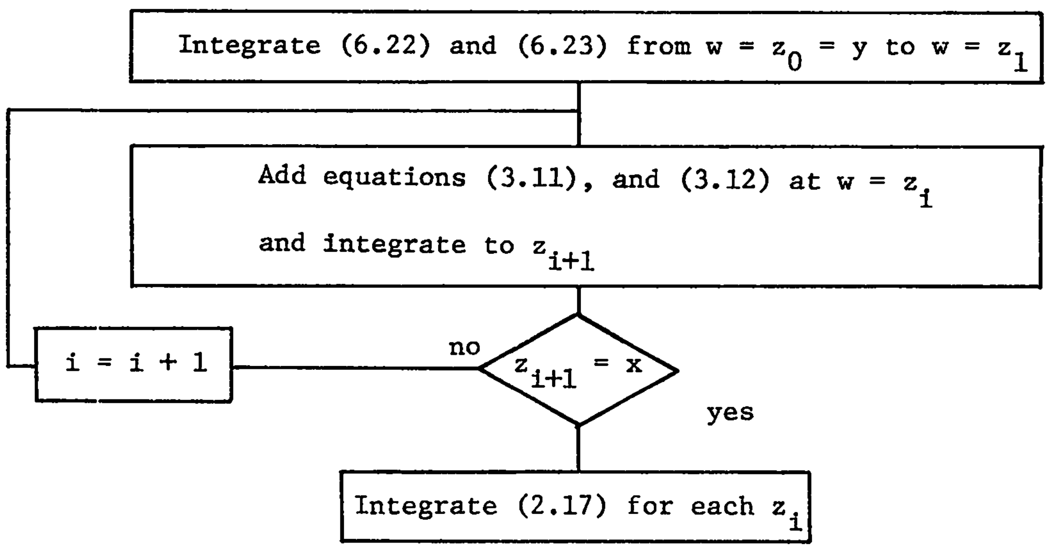

Fig. 4. Flow chart for solution of equation (1.1) when $\gamma(z)$ is a constant.

On completion the R function values are stored and the procedure is repeated for other s values. The values given in Tables 8.1, 8.2, 8.3 and 8.4 below were computed with this algorithm. In the examples, the integration was done using a fourth order Runge-Kutta-Fehlberg routine. This is the subroutine RKF given in [9]. The initial conditions (6.22c) and (6.22d) were evaluated using the subroutine QNC7 [6]. All computations were done on a CDC 7600 computer.

In order to make comparisons, a more standard solution algorithm is required. An iterative method based on the Neumann series solution of (1.1) was used. The algorithm was

$$\phi_j(z) = g(z) + \int_y^x K(z,z')\phi_{j-1}(z')\,dz' \quad ,$$

$$\phi_1(z) = g(z) \quad . \tag{8.1}$$

QNC7 was used to compute the integrals needed to evaluate the kernel $K(z,z')$. At each step in the iteration a multipoint Simpson's rule was used to evaluate the integral in (8.1). Iterations were stopped when

$$\max_{i=1,N} \left| \phi_j(z_i) - \phi_{j-1}(z_i) \right|$$

became less than a preset value.

Equation (8.1) can also be used to evaluate the R functions since, for fixed s, they satisfy integral equations of the type (1.1). It should be noted that while the imbedding moment method allows the calculation of the solution at a single point without affecting the accuracy, the iterative method must always compute the solution at a large number of points.

The first numerical experiment was to investigate the dependence of the computed solution on the number of moments used. We give several examples. In each an absolute and relative error of 10^{-10} was used to control local error in the numerical integration. In the following examples s is fixed at 0.5 and $R_R(z,x,y,s)$ is calculated for five values of z.

Example I.

$$K(z,z') = \int_0^1 e^{-5|s'|}\; e^{-|s'||z-z'|}\, ds' \quad ,$$

$$y = -1.0 \quad ,$$

$$x = 1.0 \quad ,$$

$$s = 0.5 \quad .$$

Table 8.1 gives the values of $R_R(z,1,-1,.5)$ for this problem. Fourteen moments are sufficient to obtain no further change in the tenth decimal place. This example was checked out for a total of thirty moments.

Example II.

$$K(z,z') = \int_0^1 \frac{0.1}{|s'|^{3/4}} e^{-\sin^2 s'|z - z'|} ds' \quad ,$$

$$y = -1.0 \quad ,$$

$$x = 1.0 \quad ,$$

$$s = 0.5 \quad .$$

Table 8.2 gives the results in this case. Thirteen moments were sufficient to obtain no further change in the solutions.

Example III.

$$K(z,z') = \int_0^1 e^{-s'^2/0.4}\; e^{-|s'||z-z'|}\, ds' \quad ,$$

$$y = -1.0 \quad ,$$

$$x = 1.0 \quad ,$$

$$s = 0.5 \quad .$$

TABLE 8.1

SOLUTIONS OF EXAMPLE I AS A FUNCTION OF THE NUMBER OF MOMENTS USED.

$R_R(z,1,-1,.5)$

No. of Moments	z position: -1.	-.5	0.	.5	1.
4	.68888 11719	.81584 68778	.96116 85493	1.1311 47321	1.3331 52136
5	.69045 52996	.81638 91668	.96144 57715	1.1315 55237	1.3342 32226
6	.69009 67120	.81629 04554	.96139 73096	1.1314 80924	1.3339 94584
7	.69017 25723	.81630 73167	.96140 56431	1.1314 93729	1.3340 43496
8	.69015 77211	.81630 46212	.96140 42578	1.1314 91652	1.3340 34128
9	.69016 04115	.81630 50248	.96140 44776	1.1314 91969	1.3340 35794
10	.69015 99595	.81630 49681	.96140 44446	1.1314 91923	1.3340 35519
11	.69016 00301	.81630 49756	.96140 44493	1.1314 91929	1.3340 35561
12	.69016 00198	.81630 49746	.96140 44486	.	1.3340 35555
13	.69016 00212	.81630 49748	.96140 44487	.	1.3340 35556
14	.69016 00211	.81630 49747	.	.	.
15	.	.	.	.	.
.	.	.	.		.
.	.	.	.		
.	.	.	.		.
30	.	.	.		.

TABLE 8.2

SOLUTIONS OF EXAMPLE II AS A FUNCTION OF THE NUMBER OF MOMENTS USED.

$R_R(z,1,-1,.5)$

No. of	Z POSITION				
Moments	-1.	-.5	0.	.5	1.
4	2.9232 37336	3.0646 88711	3.1742 62560	3.2526 94929	3.2984 75346
5	2.9301 20804	3.0689 13137	3.1778 63469	3.2568 43607	3.3050 79670
6	2.9288 77622	3.0682 30673	3.1772 81652	3.2561 72761	3.3038 94251
7	2.9290 79323	3.0683 30325	3.1773 67504	3.2562 70853	3.3040 85552
8	2.9290 49683	3.0683 17057	3.1773 55890	3.2562 57772	3.3040 57570
9	2.9290 53654	3.0683 18678	3.1773 57335	3.2562 59373	3.3040 61304
10	2.9290 53166	3.0683 18495	3.1773 57169	3.2562 59192	3.3040 60846
11	2.9290 53221	3.0683 18514	3.1773 57187	3.2562 59211	3.3040 60898
12	2.9290 53215	3.0683 18513	3.1773 57185	3.2562 59209	3.3040 60892
13	2.9290 53216	.	.	.	3.3040 60893
14	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
30	.	.	.	.	.

The results are given in Table 8.3. Here fifteen moments are required to reach no further change in the tenth significant figure.

It appears, at least for the examples considered so far, that the moment method converges quite rapidly. A maximum of fifteen moments was sufficient to obtain ten significant digits in all computed results.

Having examined this point we turn to the question of the accuracy of the imbedding moment method. In Table 8.4 we list selected values of R_R for the kernel of Example I, computed by the iterative method described earlier, as a function of the number of divisions used in the Simpson's rule integration. For this example we have reduced the z interval by a factor of ten from that used in Example I; specifically we treat the case

Example IV.

$$K(z,z') = \int_0^1 e^{-5|s'|} e^{-|s'||z-z'|} ds' \quad ,$$

$$y = -0.1 \quad ,$$

$$x = 0.1 \quad ,$$

$$s = 0.5 \quad .$$

We have used the smaller interval [y,x] because it is not possible in the iterative calculation to store the matrix for K(z,z') for equivalent fineness of mesh on larger intervals. It will be seen from Table 8.4 that the values obtained by the iterative method appear to be converging to the imbedding moment values as the subdivision becomes finer.

The differences in computation time are appreciable. The time required to compute the values by the iterative method ranged from 45

TABLE 8.3

SOLUTIONS OF EXAMPLE III AS A FUNCTION OF THE NUMBER OF MOMENTS USED.

$R_R(z,1,-1,.5)$

No. of Moments	z position				
	-1.	-.5	0.	.5	1.
4	5.2553 49508	5.9469 50952	6.2935 47108	6.2961 33999	5.9453 13944
5	5.4833 93477	6.1421 88547	6.4823 90312	6.4898 59094	6.1682 03452
6	5.4316 76938	6.1004 17465	6.4419 54074	6.4483 89397	6.1178 38999
7	5.4421 44348	6.1084 02892	6.4497 17355	6.4563 20748	6.1279 90174
8	5.4401 75067	6.1069 80092	6.4483 25727	6.4549 06739	6.1260 87292
9	5.4405 17524	6.1072 15033	6.4485 57049	6.4551 40368	6.1264 17082
10	5.4404 62192	6.1071 78890	6.4485 21231	6.4551 04407	6.1263 63963
11	5.4404 70526	6.1071 84087	6.4485 26412	6.4551 09580	6.1263 71941
12	5.4404 69352	6.1071 83386	6.4485 25710	6.4551 08882	6.1263 70820
13	5.4404 69507	6.1071 83475	6.4485 25799	6.4551 08971	6.1263 70968
14	5.4404 69488	6.1071 83464	6.4485 25789	6.4551 08960	6.1263 70949
15	5.4404 69490	6.1071 83465	6.4485 25790	6.4551 08961	6.1263 70951
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
30	.	.	.	.	.

TABLE 8.4

SOLUTIONS OF EXAMPLE IV FOR SEVERAL NUMBERS OF POINTS USING THE ITERATIVE METHOD.

$R_R(z,.1,-.1,.5)$

No. of Simpson Points	z position						
	-.08	-.07	-.03	.02	.03	.07	.08
41	.95266 20120	.95729 88652	.97600 58945	.99975 48996	1.0045 54216	1.0239 19308	1.0288 02958
81	.95266 20055	.95729 88587	.97600 58880	.99975 48931	1.0045 54210	1.0239 19302	1.0288 02951
121	.95266 20043	.95729 88575	.97600 58868	.99975 48919	1.0045 54209	1.0239 19300	1.0288 02950
161	.95266 20039	.95729 88571	.97600 58864	.99975 48915	1.0045 54208	1.0239 19300	1.0288 02950
201	.95266 20037	.95729 88569	.97600 58862	.99975 48913	1.0045 54208	1.0239 19300	1.0288 02950
Imbedding-moment solutions							
	.95266 20034	.95729 88566	.97600 58859	.99975 48909	1.0045 54208	1.0239 19300	1.0288 02949

seconds for 81 points to 278 seconds for 201 points. By contrast, it required less than 5.5 seconds to compute the imbedding moment values. This is not, however, a valid comparison since usually we require not the solution for the R functions, but rather for the integral equation (1.1). We will make a better comparison later.

When the kernel $K(z,z')$ is oscillatory, care must be taken when evaluating it numerically. This problem can apparently be avoided by using the imbedding moment method since the integrals required are for the initial conditions only and these may not involve oscillations. We give the following examples.

Example V.

$$K(z,z') = \int_0^1 e^{-5|s'|} e^{-i|s'||z-z'|} ds' \quad ,$$

$$y = -1.0 \quad ,$$

$$x = 1.0 \quad ,$$

$$s = 0.5 \quad .$$

Example VI.

$$K(z,z') = \int_0^1 e^{-5|s'|} e^{-i|10s'||z-z'|} ds' \quad ,$$

$$y = -1.0 \quad ,$$

$$x = 1.0 \quad ,$$

$$s = 0.5 \quad .$$

Table 8.5 gives representative results. The computation times are the times required to compute 41 z points by the imbedding moment method.

TABLE 8.5

VALUES OF $R_R(z,x,y,.5)$ FOR COMPLEX EXAMPLES V AND VI

z	Example V R_R		Example VI R_R	
	Real Part	Imaginary Part	Real Part	Imaginary Part
-1.0	0.98295 74648	-1.27015 5927	0.17997 37410	-0.11603 84534
-0.8	1.08093 1388	-1.19886 3765	-0.11028 51397	0.23978 71305
-0.6	1.16806 1621	-1.12162 2242	0.21600 17069	-0.32475 39841
-0.4	1.24373 0120	-1.03941 6319	0.89002 72187	-0.38928 04786
-0.2	1.39743 9640	-0.95325 93876	1.02228 9422	0.38787 80119
0.0	1.35881 6818	-0.86418 20522	0.29454 55710	1.02671 4905
0.2	1.39761 3741	-0.77322 10302	-0.68764 03659	0.79690 15768
0.4	1.42370 8010	-0.68140 82754	-1.08685 2831	-0.07499 61802
0.6	1.43710 1336	-0.58976 04572	-0.62413 01549	-0.56065 26009
0.8	1.43791 6727	-0.49926 89018	0.32733 57243	0.18440 11380
1.0	1.42639 4345	-0.41098 00948	-1.11974 1504	-1.04088 2114
	time 1.489 sec		time 46.131 sec	

The computation times show that we do not completely avoid the computational problems associated with an oscillatory kernel.

We now consider examples when $\gamma(z)$ is a step function. The method has been described in Section VII. Figure 5 is a general flow diagram for the program used to compute the examples. This program computes only one interior z point inside each step; this interior point could, of course, be one of the discontinuity points of $\gamma(z)$.

Example VII.

$$\phi(z) = \frac{e^{(z-1)}-1}{z-1} + \int_{-1.}^{1.} K(z,z') \, \phi(z') \, dz' \quad ,$$

$$K(z,z') = \gamma(z') \int_{0}^{1} e^{-5|s'|} \, e^{-|s'||z-z'|} \, ds' \quad , \qquad (8.2)$$

$$\gamma(z) = \begin{cases} 1. & , \quad z < 0 \\ 1.5 & , \quad z > 0 \end{cases} \qquad (8.3)$$

Note that here the function $g(z)$ is given by the transform

$$\frac{e^{(z-1)}-1}{z-1} = \int_{0}^{1} e^{-|s'||z-1|} \, ds' \quad ,$$

that is the $f(s)$ in equation (2.10) is 1 and $h(s)$ is zero.

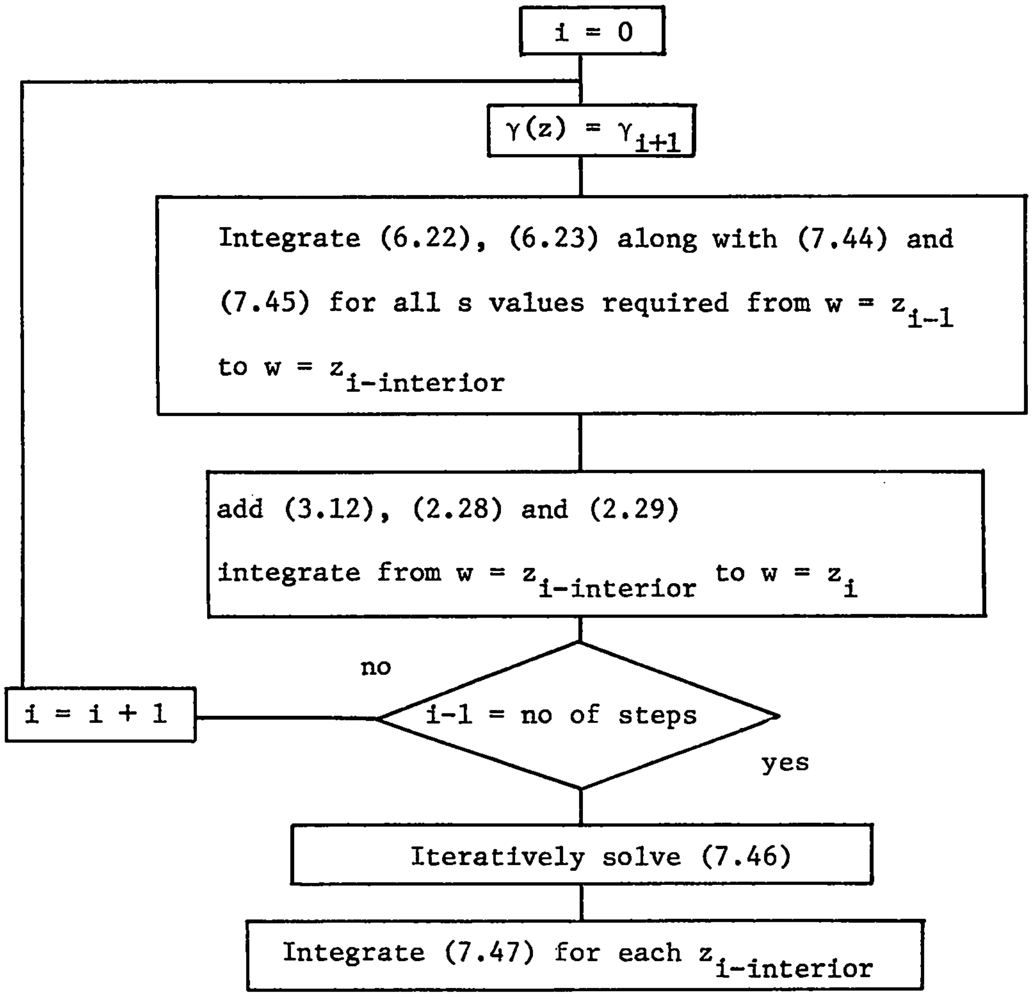

Fig. 5. Flow chart for the solution of equation (1.1) when $\gamma(z)$ is a step function.

An 81 point Simpson's rule was used to compute the iterative solution of (8.1) while a 41 point Simpson's rule was used for the iteration required to solve equations (7.16). Since the range of z is two while the range of s is only one, this should yield comparable accuracies. Table 8.6 gives the results for two internal points.

TABLE 8.6

SOLUTIONS OF (8.2) BY THE IMBEDDING-MOMENT METHOD AND BY ITERATIVE METHOD

		Imbedding-Moment	Iterative
$z =$	$(-.5)$	1.0556315	1.0556368
$z =$	$(.5)$	1.3487203	1.3487364
	time	8.368 sec	54.706 sec

The times in Table 8.6 are representative of the total procedure for a step function. If $\gamma(z)$ is a constant function the imbedding-moment method solution can be computed in about half the time while the iterative method time will remain about the same.

As a further example of the method when $\gamma(z)$ is a multiple step function we compute the solution to (8.2) with $\gamma(z)$ given by

Example VIII.

$$\gamma(z) = \begin{cases} 1. & , -1 \le z < 1.8 \\ 1.1 & , -.8 \le z < -.5 \\ 1.2 & , -.5 \le z < 0. \\ 1.3 & , 0. \le z < .4 \\ 1.4 & , 0.4 \le z < .7 \\ 1.5 & , 0.7 \le z < 1. \end{cases} \tag{8.4}$$

Table 8.7 gives the values of $\phi(z)$ for six z points computed using the imbedding-moment method, an 81 point Simpson's rule iterative method and a 241 point Simpson's rule iterative method.

TABLE 8.7

SOLUTIONS OF (8.2) WITH $\gamma(z)$ GIVEN BY (8.4).

z	$\phi(z)$ i.m.	$\phi(z)$ 81 point	$\phi(z)$ 241 point
-.9	.96128723	.96129136	.96128565
-.6	1.03355743	1.03356202	1.03355607
-.1	1.16411936	1.16412452	1.16411831
.2	1.25087242	1.25087766	1.25087140
.5	1.34621973	1.34622488	1.34621865
.8	1.45254806	1.45255298	1.45254690
Time	11.620 sec	51.177 sec	452.758sec

The examples given above in which time comparisons are made show that the imbedding moment method can offer substantial savings over the iterative method. It should be pointed out that in each comparison it was assumed that it would be necessary to integrate the kernel numerically. Since only a small fraction of all admissible kernels are integrable in closed form, this is a valid comparison. However, if the kernel can be evaluated analytically, the iterative method may be faster of the two methods.

IX REMARKS

In this dissertation we have developed a method to solve Fredholm integral equations with a special class of displacement kernels as initial value problems. The method has been shown to be both accurate and economical. One of the incidental results is the demonstration that the kernels of the reflection and transmission operators can also be obtained as the solution of initial value problems for the class of problems treated.

There remains one major unresolved question. That is: can the moment method be adapted to the general case where $\gamma(z)$ is a piecewise continuous function? It may be possible to utilize the existence of solutions for a step function $\gamma(z)$ to prove the existence for this general case. Even with this settled the problem of developing an economical integration algorithm could be formidable.

APPENDIX A

Further Examination of the Moments

This Appendix is a continuation of Section IV. We first establish two lemmas.

Lemma A.1. Under the assumptions of Lemma 4.1

$$G_0(z,x,y) = Q(z,x,x,y)/\gamma(x) \quad , \tag{A.1}$$

$$G_0(z,x,y) = Q(x,z,x,y)/\gamma(z) \quad , \tag{A.2}$$

$$H_0(z,x,y) = Q(z,y,x,y)/\gamma(y) \quad , \tag{A.3}$$

$$H_0(z,x,y) = Q(y,z,x,y)/\gamma(z) \quad . \tag{A.4}$$

Proof: Equation (A.1) follows from Lemma 4.1 and the definition of $G_0(z,x,y)$, equation (3.6). Equation A.2 then follows from A.1 by an application of the exchange property (4.1). The other two equations are proved similarly. ∎

Lemma A.2. The derivatives of the resolvent kernel of the integral equation (1.1) are given by

$$\frac{\partial^n}{\partial t^n} Q(t,r,x,y) = \left\{ \operatorname{sgn}(r-t) \right\}^n \gamma(r) \int_0^\infty a^n(s') \, k(s') \, e^{-a(s')|t-r|} \, ds'$$

$$+ \gamma(r) \int_y^x \left\{ \operatorname{sgn}(z'-t) \right\}^n Q(r,z',x,y) \, ds'$$

$$\times \int_0^\infty a^n(s') \, k(s') \, e^{-a(s')|z'-t|} \, ds \quad . \tag{A.5}$$

Proof: We use the Fredholm relation (4.7) to obtain

$$Q(t,r,x,y) = \gamma(r) \int_0^\infty k(s') \, e^{-a(s')|t-r|} \, ds'$$

$$+ \int_y^x \gamma(z') \, Q(z',r,x,y) \, dz' \int_0^\infty k(s')^{-a(s')|z'-t|} \, ds' \quad .$$

Differentiating gives

$$\frac{\partial^n}{\partial t^n} Q(t,r,x,y) = \gamma(r) \left\{\mathrm{sgn}(r - t)\right\}^n \int_0^\infty a^n(s') \, k(s') \, e^{-a(s')|t-r|} \, ds'$$

$$+ \int_y^x \gamma(z') \, Q(z',r,x,y) \, dz' \int_0^\infty a^n(s') \left\{\mathrm{sgn}(z'-t)\right\}^n$$

$$k(s') \, e^{-a(s')|z'-t|} \, ds' \quad .$$

Applying the exchange relationship to the last term, we have

$$\frac{\partial^n}{\partial t^n} Q(t,r,x,y) = \left\{\mathrm{sgn}(r - t)\right\}^n \gamma(r) \int_0^\infty a^n(s') \, k(s') \, e^{-a(s')|t-r|} \, ds'$$

$$+ \gamma(r) \int_y^x Q(r,z',x,y) \left\{\mathrm{sgn}(z' - t)\right\}^n \, dz'$$

$$\times \int_0^\infty a^n(s') \, k(s') \, e^{-a(s')|z'-t|} \, ds' \quad ,$$

which is equation (A.5). ▮

The proof of Lemma A.2 shows that the nth partial derivative of the resolvent kernel exists whenever its associated integral equation has a kernel of the form (1.1b). Since the resolvent kernel is piecewise continuous in the variables t, x and y (Lemma 2.2) and the right member of equation A.5 involves only the integral of the resolvent kernel, we can conclude that the nth partial derivatives are also piecewise continuous with respect to t, x and y.

Corollary A.3. The A and B moments are related to the resolvent kernel by the equations,

$$\frac{\partial^n}{\partial t^n} Q(t,r,x,y) \Big|_{\substack{r = y \\ t = x}} = (-1)^n \gamma(y) A_n(x,y) \quad . \tag{A.6}$$

$$\frac{\partial^n}{\partial t^n} Q(t,r,x,y) \Big|_{\substack{r = x \\ t = y}} = \gamma(x) B_n(x,y) \quad . \tag{A.7}$$

Proof: Evaluate equation (A.5) at r = y and t = x. Since sgn(y - x) = -1 and sgn(z´ - x) = -1 for all z , we have

$$\frac{\partial^n}{\partial t^n} Q(t,r,x,y) \Bigg|_{\substack{r = y \\ t = x}} = (-1)^n \gamma(y) \left\{ \int_0^\infty a^n(s') k(s') e^{-a(s')(x-y)} ds' \right.$$

$$\left. + \int_0^\infty a^n(s') k(s') ds' \int_y^x Q(y,z',x,y) e^{a(s')(z'-x)} dz' \right\}$$

$$= (-1)^n \gamma(y) \int_0^\infty a^n(s') k(s') ds' \left\{ e^{-a(s')(x-y)} \right.$$

$$\left. + \int_y^x Q(y,z',x,y) e^{a(s')(z'-x)} dz' \right\} .$$

The quantity in the curly bracket is $R_R(y,x,y,s')$, (see equation (2.15) , so

$$\frac{\partial^n}{\partial t} Q(t,r,x,y) \Bigg|_{\substack{r = y \\ t = x}} = (-1)^n \gamma(y) \int_0^\infty a^n(s') k(s') R_R(y,x,y,s') ds'$$

$$= (-1)^n \gamma(y) A_n(x,y)$$

using the definition of $A_n(x,y)$. Relation (A.7) is proved in a similar manner. ∎

Corollary A.4. The G and H moments are related to the resolvent kernel by the equations,

$$\frac{\partial^n}{\partial t^n} Q(t,r,x,y) \Big|_{t = x} = (-1)^n \gamma(r) G_n(r,x,y) \quad , \qquad \text{(A.8)}$$

$$\frac{\partial^n}{\partial t^n} Q(t,r,x,y) \Big|_{t = y} = \gamma(r) H_n(r,x,y) \quad . \qquad \text{(A.9)}$$

Proof: Since sgn $(z' - x) = \text{sgn}(r - x) = -1$, at $t = x$ equation (A.5) yields

$$\frac{\partial^n}{\partial t^n} Q(t,r,x,y) \Big|_{t = x} = (-1)^n \gamma(r) \left\{ \int_0^{\infty} a^n(s') \, k(s') \, e^{a(s')(r-x)} \, ds' \right.$$

$$\left. + \int_y^x Q(r,z',x,y) \, dz' \int_0^{\infty} a^n(s') \, k(s') \, e^{a(s')(z'-x)} \, ds' \right\}$$

$$= (-1)^n \gamma(r) \int_0^{\infty} a^n(s') \, k(s) \, ds' \left\{ e^{a(s')(r-x)} \right.$$

$$\left. + \int_y^x Q(r,z',x,y) \, e^{a(s')(z'-x)} \, dz' \right\}$$

$$= (-1)^n \gamma(r) \int_0^{\infty} a^n(s') \, k(s') \, R_R(r,x,y,s') \, ds'$$

$$= (-1)^n \gamma(r) \, G_n(r,x,y) \quad .$$

Relation (A.9) is proved similarly.∎

Corollary A.5. The C and D moments and the resolvent kernel are related by,

$$\frac{\partial^n}{\partial t^n} Q(t,r,x,y) \Big|_{\substack{r = x \\ t = x}} = (-1)^n \gamma(x) C_n(x,y) \quad , \tag{A.10}$$

$$\frac{\partial^n}{\partial t^n} Q(t,r,x,y) \Big|_{\substack{r = y \\ t = y}} = \gamma(y) D_n(x,y) \quad . \tag{A.11}$$

Proof: Equation (A.10) follows from equation (A.8) since $G_n(x,x,y) = C_n(x,y)$. Equation (A.11) is obtained from equation (A.9) by setting $r = y$, since $H_n(y,x,y) = D_n(x,y)$. ∎

We can get further insight into the relationship between the resolvent kernel and the various moments by considering the Taylor's series expansions of the resolvent kernel for special values of the arguments. It is easiest to discuss this as if y were the left edge of an imaginary slab and x the right edge. We fix x and y, and ask for the Taylor's expansion for the kernel when one of the usual kernel variables is fixed at one of the edges. For these cases the kernel is a function of a single variable and this variable, is in effect, the distance from the left or right side of the slab.

We start with Q(z,x,x,y) expanded about the left side. The Taylor's series is:

$$Q(z,x,x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial z^n} Q(z,x,x,y) \Big|_{z = y} (z - y)^n \quad .$$

Using (A.4), we find that

$$Q(z,x,x,y) = \gamma(x) \sum_{n=0}^{\infty} \frac{1}{n!} B_n(x,y)(z-y)^n \quad . \tag{A.12}$$

Using the exchange relation (4.1), we also have

$$Q(x,z,x,y) = \gamma(z) \sum_{n=0}^{\infty} \frac{1}{n!} B_n(x,y)(z-y)^n \quad . \tag{A.13}$$

If instead, we expand about the right edge, we obtain

$$Q(z,x,x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial z^n} Q(z,x,x,y) \Big|_{z=x} (z-x)^n \quad ,$$

and using (A.10) we have the two relations:

$$Q(z,x,x,y) = \gamma(x) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C_n(x,y)(z-x)^n \quad , \tag{A.14}$$

and

$$Q(x,z,x,y) = \gamma(z) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C_n(x,y)(z-x)^n \quad . \tag{A.15}$$

We can obtain similar expressions for Q(z,y,x,y) and Q(y,z,x,y). Since

$$Q(z,y,x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial z^n} Q(z,y,x,y) \Big|_{z=y} (z-y)^n \ , \tag{A.16}$$

we have

$$Q(z,y,x,y) = \gamma(y) \sum_{n=0}^{\infty} \frac{D_n(x,y)}{n!} (z - y)^n \tag{A.17}$$

and

$$Q(y,z,x,y) = \gamma(z) \sum_{n=0}^{\infty} \frac{D_n(x,y)}{n!} (z - y)^n \quad . \tag{A.18}$$

Also, since

$$Q(z,y,x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial z^n} Q(z,y,x,y) \Big|_{z = x} (z - x)^n \quad , \tag{A.19}$$

it follows that

$$Q(z,y,x,y) = \gamma(y) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n(x,y)(z - x)^n \tag{A.20}$$

and

$$Q(y,z,x,y) = \gamma(z) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n(x,y)(z - x)^n \quad . \tag{A.21}$$

The equations above also give relations between the moments in the specialized imbedding moment equations and between the moments of the imbedding moment equations. For example, using Corollary 4.2, we find that

$$G_0(z,x,y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C_n(x,y)(z - x)^n \quad , \tag{A.22}$$

$$G_0(z,x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} B_n(x,y)(z - y)^n \quad , \tag{A.23}$$

$$H_0(z,x,y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n(x,y)(z - x)^n \qquad \text{(A.24)}$$

and

$$H_0(z,x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} D_n(x,y)(z - y)^n \quad . \qquad \text{(A.25)}$$

Equating (A.12) and (A.14) we find

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C_n(x,y)(z - x)^n = \sum_{n=0}^{\infty} \frac{1}{n!} B_n(x,y)(z - y)^n \quad . \qquad \text{(A.26)}$$

Setting z = x and then z = y gives the following two relations;

$$C_0(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} B_n(x,y)(x - y)^n \quad , \qquad \text{(A.27)}$$

$$B_0(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} C_n(x,y)(x - y)^n \quad . \qquad \text{(A.28)}$$

Similarly equating (A.17) and (A.20) give

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n(x,y)(z - x)^n = \sum_{n=0}^{\infty} \frac{1}{n!} D_n(x,y)(z - y)^n \qquad \text{(A.29)}$$

and the two derived relations;

$$A_0(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} D_n(x,y)(x - y)^n \qquad \text{(A.30)}$$

and

$$D_0(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} A_n(x,y)(x - y)^n \quad . \tag{A.31}$$

In fact we can get expressions for all the moments from equations (A.26) and (A.29). Differentiating (A.26) with respect to z, and setting z = x, produces

$$(-1)^k C_k(x,y) = \sum_{n=k}^{\infty} \frac{n(n-1)\ldots(n-k+1)}{n!} B_n(x,y)(x - y)^n \quad , \tag{A.32}$$

and setting z = y produces

$$B_k(x,y) = \sum_{n=k}^{\infty} \frac{n(n-1)\ldots(n-k+1)}{n!} C_n(x,y)(x - y)^n \quad . \tag{A.33}$$

Similarly from (A.29)

$$(-1)^k A_k(x,y) = \sum_{n=k}^{\infty} \frac{n(n-1)\ldots(n-k+1)}{n!} D_n(x,y)(x - y)^n \quad , \tag{A.34}$$

$$D_k(x,y) = \sum_{n=k}^{\infty} \frac{n(n-1)\ldots(n-k+1)}{n!} A_n(x,y)(x - y)^n \quad . \tag{A.35}$$

The relations for the resolvent kernel suggest an alternate method of calculating the R functions. We start from the definition of $R_R(z,x,y,s)$,

$$R_R(z,x,y,s) = e^{a(s)(z-x)} + \int_y^x Q(z,z',x,y)\, e^{a(s)(z'-x)}\, dz \quad . \tag{2.15}$$

Evaluating equation (2.15) at z = x and using equations (A.13) and (A.15) gives

$$R_R(x,x,y,s) = 1 + \int_y^x \gamma(z') \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C_n(x,y)(z' - x)^n e^{a(s)(z'-x)} dz'$$

(A.36)

and

$$R_R(x,x,y,s) = 1 + \int_y^x \gamma(z') \sum_{n=0}^{\infty} \frac{1}{n!} B_n(x,y)(z' - y)^n e^{a(s)(z'-x)} dz' \quad .$$

(A.37)

On the other hand, evaluating equation (2.15) at $z = y$ and using expressions (A.18) and (A.21) yields

$$R_R(y,x,y,s) = e^{a(s)(y-x)} + \int_y^x \gamma(z') \sum_{n=0}^{\infty} \frac{1}{n!} D_n(x,y)(z' - y)^n e^{a(s)(z'-x)} dz'$$

(A.38)

and

$$R_R(y,x,y,s) = e^{a(s)(y-x)} + \int_y^x \gamma(z') \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n(x,y)(z' - x)^n e^{a(s)(z'-x)} dz \quad .$$

(A.39)

From the definition of $R_L(z,x,y,s)$,

$$R_L(z,x,y,s) = e^{a(s)(y-z)} + \int_y^x Q(z,z',x,y)\, e^{a(s)(y-z')} dz' \qquad (2.16)$$

we obtain in a similar fashion the four relations:

$$R_L(x,x,y,s) = e^{a(s)(y-x)} + \int_y^x \gamma(z') \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C_n(x,y)(z' - x)^n e^{a(s)(y-z')} dz' \quad , \tag{A.39}$$

$$R_L(x,x,y,s) = e^{a(s)(y-x)} + \int_y^x \gamma(z') \sum_{n=0}^{\infty} \frac{1}{n!} B_n(x,y)(z' - y)^n e^{a(s)(y-z')} dz' \quad , \tag{A.40}$$

$$R_L(y,x,y,s) = 1 + \int_y^x \gamma(z') \sum_{n=0}^{\infty} \frac{1}{n!} D_n(x,y)(z' - y)^n e^{a(s)(y-z')} dz' \quad , \tag{A.41}$$

and

$$R_L(y,x,y,s) = 1 + \int_y^x \gamma(z') \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n(x,y)(z' - x)^n e^{a(s)(y-z')} dz'. \tag{A.42}$$

If it is assumed that these series can be integrated term by term, then all the edge values of the R functions can be obtained directly from a knowledge of either the $A_n(x,y)$ and $C_n(x,y)$ or of $B_n(x,y)$ and $D_n(x,y)$ by summing a series of integrals. Since the values of the R functions are wanted for several values of s, this technique might represent a saving of computational effort.

We have carried out a number of formal operations on various power series. In Section V we have shown that these power series have at least radius of convergence X - Y. It then follows that there exists some region in which the Taylor's series for the resolvent kernel converge and in which term by term integration and differentiation are legitimate.

APPENDIX B

Derivation of The Equations for $r(z,x,s_1,s_2)$ and $\tau(z,x,s_1,s_2)$

In this appendix we derive the forward integration integro differential equations for the reflection kernel $r(z,x,s_1,s_2)$ and the transmission kernel $\tau(z,x,s_1,s_2)$. In the following we are interested in beginning the integration at $w = z$ and integrating to $w = x$.

We take as our starting point the following equation which is obtained from equation (7.9a) and the definitions of r and τ:

$$v(z,w,z,s_1) = \int_{-\infty}^{0} f(s') \, \tau(z,w,s_1,s') \, ds' + f(s_1) \, e^{a(s_1)(z-w)} + \int_{0}^{\infty} h_2(s') \, r(z,w,s_1,s') \, ds' \tag{B.1}$$

We first set

$$f(s) = 0$$

so that

$$v(z,w,z,s_1) = \int_{0}^{\infty} h_2(s') \, r(z,w,s_1,s') \, ds' \tag{B.2}$$

and

$$v_2(z,w,z,s_1) = \int_{0}^{\infty} h_2(s') \, \frac{\partial}{\partial w} r(z,w,s_1,s') \, ds' \quad . \tag{B.3}$$

We use equation (7.31) to get an expression for $v_2(z,w,z,s_1)$. This time we find from equation (7.6)

$$v_2(z,w,z,s_1) = k(s_1)\int_{-\infty}^{0} -v_1(w,w,z,s') \, ds'$$

$$\times\int_{z}^{w} \gamma(z') \, e^{a(s_1)(z-z')} R_R(z',w,z,s') \, dz'$$

$$-v_1(w,w,z,s_1) \, e^{a(s_1)(z-w)} \qquad \text{(B.4)}$$

Using (7.15)

$$v_2(z,w,z,s_1) = \int_{-\infty}^{0} \left[-v_1(w,w,z,s') \ \tau(z,w,s_1, s')\right] ds'$$

$$-v_1(w,w,z,s_1) \, e^{a(s_1)(z-w)}$$

and using the transport equations

$$v_2(z,w,z,s_1) = \int_{-\infty}^{0} \tau(z,w,s_1,s') \Big[-a(s') \ v(w,w,z,s')$$

$$+ k(s') \ \gamma(w) \ \eta(w,w,z)\Big] \, ds' + \Big[-a(s_1) \ v(w,w,z,s_1)$$

$$+ k(s_1) \ \gamma(w) \ \eta(w,w,z)\Big] \, e^{a(s_1)(z-w)} \quad . \qquad \text{(B.5)}$$

But

$$v(w,w,z,s) = f(s) = 0$$

and

$$\eta(w,w,z) = \int_{0}^{\infty} h_2(s') \ R_L(w,w,z,s') \, ds' \quad . \qquad \text{(B.6)}$$

So

$$v_2(z,w,z,s_1) = \int_{-\infty}^{0} \tau(z,w,s_1,s_2)\ k(s_2)\ \gamma(w)\ ds_2$$

$$\times \int_0^{\infty} h_2(s')\ R_L(w,w,z,s')\ ds'$$

$$+ k(s_1)\ \gamma(w)\ e^{a(s_1)(z-w)} \int_0^{\infty} h_2(s')\ R_L(w,w,z,s')\ ds'$$

$$= \int_0^{\infty} h_2(s') \Bigg[k(s_1)\ \gamma(w)\ R_L(w,w,z,s')\ e^{a(s_1)(z-w)}$$

$$+ R_L(w,w,z,s')\ \gamma(w) \int_{-\infty}^{0} k(s_2)\ \tau(z,w,s_1,s_2)\, ds_2 \Bigg] ds' \quad .$$

Putting this in equation (B.3) we find

$$\int_0^{\infty} h_2(s') \frac{\partial}{\partial w} r(z,w,s_1,s')\ ds'$$

$$= \int_0^{\infty} h_2(s') \Bigg[k(s_1)\ \gamma(w)\ R_L(w,w,z,s')\ e^{a(s_1)(z-w)}$$

$$+ R_L(w,w,z,s')\ \gamma(w) \int_{-\infty}^{0} k(s_3)\ \tau(z,w,s_1,s_3)\ ds_3 \Bigg] ds'$$

and by the usual argument

$$\frac{\partial}{\partial w} r(z,w,s_1,s_2) = R_L(w,w,z,s_2) \Bigg[\gamma(w)\ k(s_1)\ e^{a(s_1)(z-w)}$$

$$+ \gamma(w) \int_{-\infty}^{0} k(s_3)\ \tau(z,w,s_1,s_3)\ ds_3 \Bigg] \quad . \qquad \text{(B.7)}$$

To reduce this further we need relations similar to those in Lemma 7.2 for $\tau(z,w,s_1,s_2)$ but such relations do not appear to exist.

We now derive a similar equation for $\tau(z,w,s_1,s_2)$.

Again we use (7.39) but with

$$h_2(s) = 0 \quad .$$

Thus

$$v(z,w,z,s_1) = \int_{-\infty}^{0} f(s') \frac{\partial}{\partial w} \tau(z,w,s_1,s') \, ds' + f(s_1)\, a(s_1)\, e^{a(s_1)(z-w)} , \tag{B.8}$$

$$v_2(z,w,z,s_1) = \int_{-\infty}^{0} f(s') \frac{\partial}{\partial w} \tau(z,w,s_1,s') \, ds' - f(s_1) a(s_1)\, e^{a(s_1)(z-w)} . \tag{B.9}$$

The expression for $v_2(z,w,z,s_1)$ is obtained as before and is equation (7.56), however, we now have

$$v(w,w,z,s) = f(s)$$

$$v(w,w,z,s) = \int_{-\infty}^{0} f(s')\, R_R(w,w,z,s') \, ds' . \tag{B.10}$$

So

$$v_2(z,w,z,s_1) = \int_{-\infty}^{0} \tau(z,w,s_1,s') \Bigg[-a(s')\, f(s') + k(s')\, \gamma(w) \int_{-\infty}^{0} f(s_2)\, R_R(w,w,z,s_2)\, ds_2 \Bigg] ds'$$
$$- a(s_1)\, f(s_1)\, e^{a(s_1)(z-w)}$$
$$+ k(s_1)\, \gamma(w)\, e^{a(s_1)(z-w)} \int_{-\infty}^{0} f(s_2)\, R_R(w,w,z,s_2)\, ds_2$$

or

$$v_2(z,w,z,s_1) = -a(s_1)\, f(s_1)\, e^{a(s_1)(z-w)}$$

$$+\int_{-\infty}^{0} f(s') \Big[-a(s')\, \tau(z,w,s_1,s\ )$$

$$+ \gamma(w)\, R_R(w,w,z,s') \int_{-\infty}^{0} k(s_2)\, \tau(z,w,s_1,s_2)\, ds_2$$

$$+ \gamma(w)\, R_R(w,w,z,s')\, k(s_1)\, e^{a(s_1)(z-w)} \Big] ds'$$

Putting this in (B.9), canceling the non-integrated term and using the usual argument leads to

$$\frac{\partial}{\partial w} \tau(z,w,s_1,s_2) = -a(s_2)\, \tau(z,w,s_1,s_2)$$

$$+ \gamma(w)\, R_R(w,w,z,s_2) \Big[k(s_1)\, e^{a(s_1)(z-w)}$$

$$+ \int_{-\infty}^{0} k(s_3)\, \tau(z,w,s_1,s_3) \Big] ds_3 \qquad \text{(B.11)}$$

The same integral is seen to occur in this equation as appeared in (B.7). A search for exchange relations for the transmission kernels similar to those given in Lemma 7.2 was unsuccessful.

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